Spectral Theory for the Hyperbolic Plane

In our discussion of spectral theory we naturally start with the hyperbolic plane itself, the primary example of a hyperbolic surface of infinite area. In this section we will study the Laplacian on \( \mathbb{H} \),

\[
\Delta_{\mathbb{H}} = -y^2(\partial^2_x + \partial^2_y).
\]

Not surprisingly, in this case we can write explicit formulas for the basic spectral objects.

For \( s \in \mathbb{C} \) the function \( y^s \) satisfies an eigenvalue equation,

\[
\Delta_{\mathbb{H}} y^s = s(1-s)y^s. \tag{4.1}
\]

If \( \text{Re} s = \frac{1}{2} \), then by multiplying \( y^s \) by appropriate cutoff functions we can construct an orthonormal sequence \( \{\phi_n\} \) such that \( \| (\Delta_{\mathbb{H}} - \lambda) \phi_n \| \to 0 \) for \( \lambda = s(1-s) \in [1/4, \infty) \). (The details of this construction can be found later, in the proof of Proposition 7.2.) Weyl’s criterion (Theorem A.13) then shows that \([1/4, \infty)\) is contained in the essential spectrum.

We will get a finer picture below by explicit formulas for the resolvent and spectral projections. It turns out (Theorem 4.2) that \([1/4, \infty)\) is the full spectrum of \( \Delta_{\mathbb{H}} \) and is absolutely continuous.

### 4.1 Resolvent

The resolvent of a positive self-adjoint operator \( A \) is the bounded operator \( (A - z)^{-1} \), defined for \( z \in \mathbb{C} - [0, \infty) \) by the spectral theorem. In the case of the operator \( \Delta_{\mathbb{H}} \), equation (4.1) hints at the fact that it will be convenient to substitute \( z = s(1-s) \) and use \( s \) as our spectral parameter.

**Definition.** For \( \text{Re} s > \frac{1}{2}, \ s \notin \left[ \frac{1}{2}, 1 \right] \), the *resolvent* of \( \Delta_{\mathbb{H}} \) is defined by

\[
R_{\mathbb{H}}(s) := (\Delta_{\mathbb{H}} - s(1-s))^{-1}.
\]
The resolvent kernel $R_\mathbb{H}(s; z, z')$ is the integral kernel of $R_\mathbb{H}(s)$, with respect to the hyperbolic area element $dA = y^{-2} dx dy$. With respect to $dA$, the integral kernel of $I$ is written $y^2 \delta(z - z')$, and so the equation

$$(\Delta_\mathbb{H} - s(1 - s))R_\mathbb{H}(s) = I$$

becomes

$$(\Delta_\mathbb{H} - s(1 - s))R_\mathbb{H}(s; z, z') = y^2 \delta(z - z'),$$

with $\Delta_\mathbb{H}$ acting on the $z$ coordinate. This shows that the resolvent kernel is essentially the classical Green’s function for the operator $(\Delta_\mathbb{H} - s(1 - s))$. The symmetry of $\mathbb{H}$ implies that the Green’s function depends only on hyperbolic distance, so we can write $R_\mathbb{H}(s; z, z') = f_s(d(z, z'))$ for some function $f_s$. To translate (4.2) into an equation for $f_s$, we switch to geodesic polar coordinates $(r, \theta)$ centered on $z'$, setting $r = d(z, z')$. In (2.10) we noted that the geodesic polar form of the hyperbolic metric is $ds^2 = dr^2 + \sinh^2 r d\theta^2$. The corresponding Laplacian is

$$\Delta_\mathbb{H} = -\frac{1}{\sinh r} \hat{\partial}_r (\sinh r \hat{\partial}_r) - \frac{1}{\sinh^2 r} \hat{\partial}_\theta^2.$$ 

Thus the homogeneous equation corresponding to (4.2) is

$$\begin{bmatrix} -\frac{1}{\sinh r} \hat{\partial}_r (\sinh r \hat{\partial}_r) - s(1 - s) \end{bmatrix} f_s(r) = 0, \quad (4.3)$$

for $r > 0$.

To solve (4.3), we make one further transformation by setting $g_s(\sigma) = f_s(r)$, where $\sigma := \cosh^2(r/2)$. Then (4.3) becomes

$$\sigma (1 - \sigma) g''_s + (1 - 2\sigma) g'_s - s(1 - s) g_s = 0,$$  

for $\sigma > 1$. This is an improvement because equation (4.4) is a special case of the classical hypergeometric equation

$$z(1 - z)h''(z) + (c - (a + b + 1)z)h'(z) - ab h(z) = 0.$$  

The standard solution is the (Gauss) hypergeometric function, defined for $|z| < 1$ by the series

$$F(a, b; c; z) := 1 + \frac{ab}{c} z + \frac{a(a + 1)b(b + 1)}{2! c(c + 1)} z^2 + \cdots.$$  

(This is also commonly denoted by $\, _2F_1$, where the subscripts refer to the numbers of parameters of each type.) The solution $F(a, b; c; z)$ is regular at $z = 0$, whereas we require a solution regular at $z = \infty$ since $\sigma \in [1, \infty)$. From the Kummer set of solutions (see, e.g., [1, equation (15.5.7)]) we select

$$h(z) = z^{-a} F(a, a + 1 - c; a + 1 - b; z^{-1}).$$

Based on (4.4), we set $a = s$, $b = 1 - s$, and $c = 1$, so that our proposed solution is