In this chapter we’ll develop explicit formulas for the resolvent kernels of the other elementary surfaces: the hyperbolic and parabolic cylinders. These explicit formulas will serve as building blocks when we turn to the construction of the resolvent in the general case in Chapter 6. This is because of the decomposition result of Theorem 2.13, which showed that the ends of nonelementary hyperbolic surfaces are funnels and cusps.

5.1 Hyperbolic cylinders

Recall from Section 2.4 the basic model for a hyperbolic cylinder:

\[ C_{\ell} := \Gamma_{\ell} \backslash \mathbb{H}, \quad \Gamma_{\ell} := \langle z \mapsto e^\ell z \rangle. \]

As corresponding fundamental domain we will use \( \mathcal{F}_{\ell} := \{ 1 \leq |z| \leq e^\ell \} \). This model allows us to conveniently write functions on \( C_{\ell} \) in terms of their lifts to \( \mathbb{H} \).

![Hyperbolic cylinder coordinates.](image)

We can also introduce a natural set of geodesic coordinates on \( C_{\ell} \), based on distance from the central closed geodesic. Let \( t \in \mathbb{R}/\ell\mathbb{Z} \) be an arc-length parameter for the central geodesic. For each value of \( t \), we introduce a longitudinal geodesic
intersecting the central geodesic orthogonally, and let \( r \) denote the signed distance from the central geodesic along these longitudinal geodesics. (Geodesic coordinates constructed in this way are usually called Fermi coordinates.)

Figure 5.1 illustrates the two coordinate models for the hyperbolic cylinder. If we identify the central geodesic \( r = 0 \) with the \( y \)-axis and the longitudinal geodesic \( t = 0 \) with the arc of the unit circle, then the explicit relation between the coordinates is

\[
z = e^t \frac{e^r + i}{e^r - i}.
\]

We can use this covering map to derive the form of the metric in the \((r, t)\) coordinates,

\[
ds^2 = dr^2 + \cosh^2 r \, dt^2.
\]  

(One could also deduce this from the assumptions on \( r \) and \( t \) and the curvature condition (2.9).)

We will give two separate approaches to construction of the resolvent kernel for \( C_\ell \). In the first, we construct the lift of the kernel to \( \mathbb{H} \) by averaging \( R_\mathbb{H}(s; z, w) \) over the action of \( \Gamma_\ell \). In the second, we will solve the Green’s function equation directly in the geodesic coordinates.

In Section 4.1 we found that

\[
R_\mathbb{H}(s; z, z') = g_s(\sigma(z, z')),
\]

where \( \sigma(z, z') = \cosh^2(d(z, z')/2) \) and \( g_s(\sigma) \sim c_s \sigma^{-s} \) as \( \sigma \to \infty \). We thus have

\[
R_\mathbb{H}(s; z, e^{k\ell}z') = O(e^{-s|k|\ell}),
\]

as \( k \to \pm \infty \), uniformly for \( z, z' \) in compact sets. Hence the sum

\[
R_{C_\ell}(s; z, z') := \sum_{k \in \mathbb{Z}} R_\mathbb{H}(s; z, e^{k\ell}z')
\]  

(5.2)

converges to an analytic function of \( s \) for \( \text{Re} \, s > 0 \). This already demonstrates the analytic continuation of \( R_{C_\ell}(s; z, w) \) across the continuous spectrum \( \text{Re} \, s = \frac{1}{2} \). For continuation further to the left, we’ll follow the argument by Guillopé [83].

Given the definition (4.12) of \( g_s(\sigma) \), the hypergeometric series (4.6) yields the expansion

\[
g_s(\sigma) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{\Gamma(s + n)^2}{n!\Gamma(2s + n)} \sigma^{-s-n},
\]

for \( \sigma > 1 \) and \( s \notin -\mathbb{N}_0 \). We can truncate the sum after finitely many terms to obtain

\[
g_s(\sigma) = \sum_{n=0}^{N-1} \frac{\Gamma(s + n)^2}{4\pi n!\Gamma(2s + n)} \sigma^{-s-n} + F_N(s, \sigma),
\]  

(5.3)

with \( F_N(s, \sigma) \) analytic in \( \text{Re} \, s > -N \) and satisfying

\[
F_N(s, \sigma) = O(\sigma^{-s-N}) \quad \text{as} \quad \sigma \to \infty.
\]  

(5.4)