A Crash Course in Several Complex Variables

Prologue: The function theory of several complex variables (SCV) is—obviously—a generalization of the subject of one complex variable. Certainly some of the results in the former subject are inspired by ideas from the latter subject. But SCV really has an entirely new character.

One difference (to be explained below) is that in one complex variable every domain is a domain of holomorphy; but in several complex variables some domains are and some are not (this is a famous theorem of F. Hartogs). Another difference is that the Cauchy–Riemann equations in several-variables form an overdetermined system; in one variable they do not. There are also subtleties involving the $\partial$-Neumann boundary value problem, such as subellipticity; we cannot treat the details here, but see [KRA3].

Most of the familiar techniques of one complex variable—integral formulas, Blaschke products, Weierstrass products, the argument principle, conformality—either do not exist or at least take a quite different form in the several-variable setting. New tools, such as sheaf theory and the $\partial$-Neumann problem, have been developed to handle problems in this new setting.

Several complex variables is exciting for its interaction with differential geometry, with partial differential equations, with Lie theory, with harmonic analysis, and with many other parts of mathematics. In this text we shall see several complex variables lay the groundwork for a new type of harmonic analysis.

Next we turn our attention to the function theory of several complex variables. One of the main purposes of this book is to provide a foundational introduction to the harmonic analysis of several complex variables. So this chapter constitutes a transition. It will provide the reader with the basic language and key ideas of several complex variables so that the later material makes good sense.

As a working definition let us say that a function $f(z_1, z_2, \ldots, z_n)$ of several complex variables is holomorphic if it is holomorphic in each variable separately.
We shall say more about different definitions of holomorphicity, and their equivalence, as the exposition progresses.

5.1 What Is a Holomorphic Function?

**Capsule:** There are many ways to define the notion of holomorphic function of several complex variables. A function is holomorphic if it is holomorphic (in the classical sense) in each variable separately. It is holomorphic if it satisfies the Cauchy–Riemann equations. It is holomorphic if it has a local power series expansion about each point. There are a number of other possible formulations. We explore these, and some of the elementary properties of holomorphic functions, in this section.

In the discussion that follows, a *domain* is a connected, open set. Let us restrict attention to functions $f : \Omega \to \mathbb{C}$, $\Omega$ a domain in $\mathbb{C}^n$, that are locally integrable (denoted $f \in L^1_{\text{loc}}$). That is, we assume that $\int_K |f(z)| \, dV(z) < \infty$ for each compact subset $K$ of $\Omega$. In particular, in order to avoid aberrations, we shall only discuss functions that are distributions (to see what happens to function theory when such a standing hypothesis is not enforced, see the work of E.R. Hedrick [HED]). Distribution theory will not, however, play an explicit role in what follows.

For $p \in \mathbb{C}^n$ and $r \geq 0$, we let

$$D^n (p, r) = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |p_j - z_j| < r \text{ for all } j \}$$

and

$$\overline{D^n} (p, r) = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : |p_j - z_j| \leq r \text{ for all } j \}.$$  

These are the open and closed *polydisks* of radius $r$.

We also define balls in complex space by

$$B(z, r) = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_j |p_j - z_j|^2 < r^2 \right\}$$

and

$$\overline{B}(z, r) = \left\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_j |p_j - z_j|^2 \leq r^2 \right\}.$$  

We now offer three plausible definitions of holomorphic function on a domain $\Omega \subseteq \mathbb{C}^n$:

**DEFINITION A:** A function $f : \Omega \to \mathbb{C}$ is holomorphic if for each $j = 1, \ldots, n$ and each fixed $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$ the function

$$\zeta \mapsto f(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)$$

satisfies

$$\int_K |f(z)| \, dV(z) < \infty$$

for each compact subset $K$ of $\Omega$. In particular, in order to avoid aberrations, we shall only discuss functions that are distributions (to see what happens to function theory when such a standing hypothesis is not enforced, see the work of E.R. Hedrick [HED]). Distribution theory will not, however, play an explicit role in what follows.