Pseudoconvexity and Domains of Holomorphy

Prologue: Pseudoconvexity and domains of holomorphy are the two fundamental ideas in the function theory of several complex variables. The first is a differential geometric condition on the boundary. The second is an idea that comes strictly from function theory. The big result in the subject—the solution of the Levi problem—is that these two conditions are equivalent.

It requires considerable machinery to demonstrate the solution of the Levi problem, and we cannot present all of it here. What we can do is to describe all the key ingredients. And this is important, for these ingredients make up the fundamental tools in the subject.

6.1 Comparing Convexity and Pseudoconvexity

Capsule: As indicated in the previous section, convexity and pseudoconvexity are closely related. Pseudoconvexity is, in a palpable sense, a biholomorphically invariant version of convexity. In this section we explore the connections between convexity and pseudoconvexity.

A straightforward calculation (see [KRA4]) establishes the following result:

Prelude: The next proposition demonstrates, just as we did above for strong convexity, that strong pseudoconvexity is a stable property under $C^2$ perturbations of the boundary. This is one of the main reasons that strong pseudoconvexity is so useful.

**Proposition 6.1.1** If $\Omega$ is strongly pseudoconvex then $\Omega$ has a defining function $\tilde{\rho}$ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \overline{z}_k} (P) w_j \overline{w}_k \geq C|w|^2$$

for all $P \in \partial \Omega$, all $w \in \mathbb{C}^n$.
By continuity of the second derivatives of \( \tilde{\rho} \), the inequality in the proposition must in fact persist for all \( z \) in a neighborhood of \( \partial \Omega \). In particular, if \( P \in \partial \Omega \) is a point of strong pseudoconvexity then so are all nearby boundary points.

**Example 6.1.2** Let \( \Omega = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^4 < 1 \} \). Then \( \rho(z_1, z_2) = -1 + |z_1|^2 + |z_2|^4 \) is a defining function for \( \Omega \) and the Levi form applied to \( (w_1, w_2) \) is \( |w_1|^2 + 4|z_2|^2|w_2|^2 \). Thus \( \partial \Omega \) is strongly pseudoconvex except at boundary points where \( |z_2|^2 = 0 \) and the tangent vectors \( w \) satisfy \( w_1 = 0 \). Of course these are just the boundary points of the form \( (e^{i\theta}, 0) \). The domain is (weakly) Levi pseudoconvex at these exceptional points.

Pseudoconvexity describes something more (and less) than classical geometric properties of a domain. However, it is important to realize that there is no simple geometric description of pseudoconvex points. Weakly pseudoconvex points are far from being well understood at this time. Matters are much clearer for strongly pseudoconvex points:

**Prelude:** One of the biggest unsolved problems in the function theory of several complex variables is to determine which pseudoconvex boundary points may be “convexified”—in the sense that there is a biholomorphic change of coordinates that makes the point convex in some sense. We see next that, for a strongly pseudoconvex point, matters are quite simple.

**Lemma 6.1.3 (Narasimhan)** Let \( \Omega \subset \subset \mathbb{C}^n \) be a domain with \( C^2 \) boundary. Let \( P \in \partial \Omega \) be a point of strong pseudoconvexity. Then there is a neighborhood \( U \subseteq \mathbb{C}^n \) of \( P \) and a biholomorphic mapping \( \Phi \) on \( U \) such that \( \Phi(U \cap \partial \Omega) \) is strongly convex.

**Proof:** By Proposition 6.1.1 there is a defining function \( \tilde{\rho} \) for \( \Omega \) such that

\[
\sum_{j,k} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial z_k}(P)w_j \overline{w_k} \geq C|w|^2
\]

for all \( w \in \mathbb{C}^n \). By a rotation and translation of coordinates, we may assume that \( P = 0 \) and that \( \nu = (1, 0, \ldots, 0) \) is the unit outward normal to \( \partial \Omega \) at \( P \). The second-order Taylor expansion of \( \tilde{\rho} \) about \( P = 0 \) is given by

\[
\tilde{\rho}(w) = \tilde{\rho}(0) + \sum_{j=1}^{n} \frac{\partial \tilde{\rho}}{\partial z_j}(P)w_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial z_k}(P)w_j w_k \\
+ \sum_{j=1}^{n} \frac{\partial \tilde{\rho}}{\partial \overline{z}_j}(P)\overline{w}_j \overline{w}_k + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial \overline{z}_j \partial z_k}(P)\overline{w}_j \overline{w}_k \\
+ \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial \overline{z}_j \partial z_k}(P)w_j \overline{w}_k + o(|w|^2)
\]