Chapter 7 dealt with systems of functions generated by integer-translates of a single function in $L^2(\mathbb{R})$. We will now generalize this setup and consider translates of a given countable family of functions rather than just one function. Such systems of functions are called shift-invariant systems. Our goal is to characterize various frame properties for shift-invariant systems, a subject that was treated first in the paper [58] by Ron and Shen. The presentation is inspired by the approach by Janssen in [44]. The derived results will play an important role in the analysis of Gabor systems in Chapter 9.

The theory for shift-invariant systems is based on two classes of operators on $L^2(\mathbb{R})$, namely,

Translation by $a \in \mathbb{R}$, $T_a : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $(T_a f)(x) = f(x - a)$;  
Modulation by $b \in \mathbb{R}$, $E_b : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $(E_b f)(x) = e^{2\pi ibx} f(x)$.

Both classes of operators were introduced in Section 2.9; in particular, we will use their interaction with the Fourier transform, a subject that is also treated in Section 2.9.

8.1 Frame-properties of shift-invariant systems

Let $\{g_m\}_{m \in I}$ be a countable collection of functions in $L^2(\mathbb{R})$ and $a > 0$ be a given (shift-) parameter. The shift-invariant system generated by $\{g_m\}_{m \in I}$ and $a$ is the collection of functions $\{g_m(\cdot - na)\}_{m \in I, n \in \mathbb{Z}}$. Formulated in
terms of the translation operator, the system has the form \( \{ T_{na} g_m \}_{m \in I, n \in \mathbb{Z}} \). Usually, we will let \( I = \mathbb{Z} \), in which case we simply write

\[
\{ g_{nm} \} := \{ T_{na} g_m \}_{m, n \in \mathbb{Z}}. \tag{8.1}
\]

As already mentioned, our goal is to characterize various frame properties for systems of the form \( \{ g_{nm} \} \). The Fourier transform will be an important tool; in fact, the characterizations will be formulated in terms of certain conditions on the functions \( \hat{g}_m \).

In particular, we will present equivalent conditions for two systems \( \{ g_{nm} \} \) and \( \{ h_{nm} \} \) to form dual frames. Given the two shift-invariant Bessel systems \( \{ g_{nm} \} \) and \( \{ h_{nm} \} \), and two functions \( e, f \in L^2(\mathbb{R}) \), the analysis of the function \( \rho(e, f) \) defined by

\[
\rho(e, f) : \mathbb{R} \to \mathbb{C}, \quad \rho(e, f)(x) = \sum_{m, n \in \mathbb{Z}} \langle T_x e, g_{nm} \rangle \langle h_{nm}, T_x f \rangle \tag{8.2}
\]

will play a central role. The reason for considering this function is apparent from our discussion of general dual frame pairs in Section 5.7: in fact, Lemma 5.7.1 shows that two Bessel sequences \( \{ g_{nm} \} \) and \( \{ h_{nm} \} \) form dual frames for \( L^2(\mathbb{R}) \) if and only if

\[
\rho(e, f)(0) = \langle e, f \rangle, \quad \forall e, f \in L^2(\mathbb{R}).
\]

In the first result, we will derive a useful consequence of the Bessel condition.

**Lemma 8.1.1** Assume that \( \{ g_{nm} \} \) is a Bessel sequence with bound \( B \). Then

\[
\sum_{m \in \mathbb{Z}} |\hat{g}_m(\nu)|^2 \leq aB, \quad a.e. \ \nu \in \mathbb{R}. \tag{8.3}
\]

**Proof.** Let \( f \in L^2(\mathbb{R}) \), and consider the function

\[
\rho(f, f)(x) = \sum_{m, n \in \mathbb{Z}} |\langle T_x f, g_{nm} \rangle|^2; \tag{8.4}
\]

it corresponds to the general expression in (8.2) in the case \( h_m = g_m \). The assumption that \( \{ g_{nm} \} \) is a Bessel sequence with bound \( B \) implies that \( \rho(f, f) \) is bounded: in fact,

\[
\rho(f, f)(x) \leq B \|T_x f\|^2 = B \|f\|^2, \quad \forall x \in \mathbb{R}. \tag{8.5}
\]

The shift-invariance of the system \( \{ g_{nm} \} \) implies that \( \rho(f, f) \) is periodic with period \( a \) (Exercise 8.1), so we can consider its Fourier expansion in