Gabor Bases and Frames

In this chapter we will consider the construction and properties of the class of Gabor frames for the Hilbert space $L^2(\mathbb{R})$. The analysis and application of Gabor systems is one part of the field of time-frequency analysis, which is more broadly explored in Gröchenig’s text [Grö01].

In Chapter 10 we focused on systems of weighted exponentials $\{e^{2\pi inx}\}_{n \in \mathbb{Z}}$ and systems of translates $\{g(x - k)\}_{k \in \mathbb{Z}}$. Each of these systems is generated by applying a single type of operation (modulation or translation) to a single generating function ($\varphi$ or $g$). The resulting sequences have many applications, but their closed spans can only be proper subspaces of $L^2(\mathbb{R})$. In contrast, Gabor systems incorporate both modulations and translations, and can be frames for all of $L^2(\mathbb{R})$.

Gabor systems were briefly introduced in Example 8.10 and are defined precisely as follows.

**Definition 11.1.** A lattice Gabor system, or simply a Gabor system for short, is a sequence in $L^2(\mathbb{R})$ of the form $G(g,a,b) = \{e^{2\pi ibnx}g(x - ak)\}_{k,n \in \mathbb{Z}}$, where $g \in L^2(\mathbb{R})$ and $a, b > 0$ are fixed. We call $g$ the generator or the atom of the system, and refer to $a, b$ as the lattice parameters.

More generally, an “irregular” Gabor system is a sequence of the form $G(g, \Lambda) = \{e^{2\pi ibx}g(x - a)\}_{(a,b) \in \Lambda}$, where $\Lambda$ is an arbitrary countable set of points in $\mathbb{R}^2$. Lattice Gabor systems have many attractive features and applications, and are much easier to analyze than irregular Gabor systems, so we focus on lattice systems for most of this chapter. For more details on irregular Gabor systems, we refer to [Grö01] or the survey paper [Hei07].

We are especially interested in Gabor systems that form frames or Riesz bases for $L^2(\mathbb{R})$. Naturally, if $G(g,a,b)$ is a frame for $L^2(\mathbb{R})$, then we call it a Gabor frame, and if it is a Riesz basis, then we call it a Gabor Riesz basis or an exact Gabor frame.
Gabor systems are named after Dennis Gabor (1900–1979), who was awarded the Nobel prize for his invention of holography. In his paper [Gab46], Gabor proposed using the Gabor system $\mathcal{G}(\phi, 1, 1)$ generated by the Gaussian function $\phi(x) = e^{-\pi x^2}$. Von Neumann [vN32, p. 406] had earlier claimed (without proof) that $\mathcal{G}(\phi, 1, 1)$ is complete in $L^2(\mathbb{R})$, i.e., its finite linear span is dense. Gabor conjectured (incorrectly, as we will see) that every function in $L^2(\mathbb{R})$ could be represented in the form

$$f = \sum_{k,n \in \mathbb{Z}} c_{kn}(f) M_n T_k \phi$$

(11.1)

for some scalars $c_{kn}(f)$; see [Gab46, Eq. 1.29]. This is one reason why general families $\mathcal{G}(g, a, b)$ are named in his honor (see [Jan01] for additional historical remarks and references).

Von Neumann’s claim of completeness was proved in [BBGK71], [Per71], and [BGZ75]. However, completeness is a weak property and does not imply the existence of expansions of the form given in equation (11.1). Reading a bit extra into what von Neumann and Gabor actually wrote, possibly they expected that $\mathcal{G}(\phi, 1, 1)$ would be a Schauder basis or a Riesz basis for $L^2(\mathbb{R})$. In fact, $\mathcal{G}(\phi, 1, 1)$ is neither, as it is overcomplete in the sense that any single element may be removed and still leave a complete system. In fact, the excess is precisely 1, because this system becomes incomplete as soon as two elements are removed. However, even with one element removed, the resulting exact system forms neither a Schauder basis nor a Riesz basis; cf. [Fol89, p. 168]. In fact, Janssen proved in [Jan81] that Gabor’s conjecture that each $f \in L^2(\mathbb{R})$ has an expansion of the form in equation (11.1) is true, but he also showed that the series converges only in the sense of tempered distributions—not in the norm of $L^2$—and the coefficients $c_{kn}$ grow with $k$ and $n$ (see also [LS99]).

Today we realize that there are no “good” Gabor Riesz bases $\mathcal{G}(g, a, b)$ for $L^2(\mathbb{R})$. Indeed, the Balian–Low Theorem, which we mentioned in Chapter 8 and will consider in detail in Section 11.8, implies that only “badly behaved” atoms $g$ can generate Gabor Riesz bases. On the other hand, redundant Gabor frames with nice generators do exist, and they provide us with useful tools for many applications. We will study the construction and special properties of Gabor frames in this chapter.

### 11.1 Time-Frequency Shifts

We recall the following operations on functions $f : \mathbb{R} \rightarrow \mathbb{C}$.

- **Translation:** $(T_a f)(x) = f(x - a), \quad a \in \mathbb{R}$.
- **Modulation:** $(M_b f)(x) = e^{2\pi ibx} f(x), \quad b \in \mathbb{R}$.
- **Dilation:** $(D_r f)(x) = r^{1/2} f(rx), \quad r > 0$. 