On the Geometry of Cohomogeneity One Manifolds with Positive Curvature

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Abstract We discuss manifolds with positive sectinal curvature on which a group acts isometrically with one dimensional quotient. A number of the known examples have this property, but some potential families for new examples in dimension 7 arise as well. We discuss the geometry of these known examples and the connection that the candidates have with self-dual Einstein metrics.

There are very few known examples of manifolds with positive sectional curvature. Apart from the compact rank one symmetric spaces, they exist only in dimensions 24 and below and are all obtained as quotients of a compact Lie group equipped with a biinvariant metric under an isometric group action. They consist of certain homogeneous spaces in dimensions 6, 7, 12, 13, and 24 due to Berger [Be], Wallach [Wa], and Aloff–Wallach [AW], and of biquotients in dimensions 6, 7, and 13 due to Eschenburg [E1],[E2],[E3] and Bazaikin [Ba].

When trying to find new examples, it is natural to search among manifolds with large isometry group, a program initiated by K. Grove in the 1990s, see [Wi] for a recent survey. Homogeneous spaces with positive curvature were classified in [Be],[Wa],[BB] in the 1970s. The next natural case to study is therefore manifolds on which a group acts isometrically with one dimensional quotient, so called cohomogeneity one manifolds. L. Verdiani [V1, V2] showed that in even dimensions, positively curved cohomogeneity one manifolds are equivariantly diffeomorphic to an isometric action on a rank one symmetric space. In odd dimensions, K. Grove and the author observed in 1998 that there are infinite families among the known nonsymmetric positively curved manifolds that admit isometric cohomogeneity one actions, and suggested a family of potential 7 dimensional candidates $P_k$. In [GWZ], a classification in odd dimensions was carried out, and another family $Q_k$ and an isolated manifold $R$ emerged in dimension 7. It is not yet known whether these manifolds admit a cohomogeneity one metric with positive curvature, although they all admit one with nonnegative curvature as a consequence of the main result in [GZ].
In [GWZ], the authors also discovered an intriguing connection that the manifolds \( P_k \) and \( Q_k \) have with a family of self-dual Einstein orbifold metrics constructed by Hitchin [Hi1] on \( S^4 \). They naturally give rise to 3-Sasakian metrics on \( P_k \) and \( Q_k \), which by definition have lots of positive curvature already.

The purpose of this survey is threefold. In Section 2, we study the positively curved cohomogeneity one metrics on known examples with positive curvature, including the explicit functions that define the metric. In Section 3, we describe the classification theorem in [GWZ]. It is remarkable that among 7-manifolds where \( G = S^3 \times S^3 \) acts by cohomogeneity one, one has the known positively curved Eschenburg spaces \( E_p \), the Berger space \( B_7 \), the Aloff–Wallach space \( W_7 \), and the sphere \( S^7 \), and that the candidates \( P_k, Q_k \), and \( R \) all carry such an action as well.

We thus carry out the proof in this most intriguing case where \( G = S^3 \times S^3 \) acts by cohomogeneity one on a compact 7-dimensional simply connected manifold. In Section 4, we describe the relationship to Hitchin’s self-dual Einstein metrics. We also discuss some curvature properties of these Einstein metrics and the metrics they define on \( P_k \) and \( Q_k \). The behavior of these metrics, as well as the known metrics with positive curvature, are illustrated in a series of pictures.

1 Preliminaries

In this section, we discuss the basic structure of cohomogeneity one actions and the significance of the Weyl group. For more details, we refer the reader to [AA, Br, GZ, Mo]. We assume from now on that the manifold \( M \) and the group \( G \) that acts on \( M \) are compact and will only consider the most interesting case, where \( M/G = I = [0, L] \). If \( \pi: M \to M/G \) is the orbit projection, the inverse images of the interior points are the regular orbits and \( B^- = \pi^{-1}(0) \) and \( B^+ = \pi^{-1}(L) \) are the two nonregular orbits. Choose a point \( x_- \in B_- \) and let \( \gamma: [0, L] \to M \) be a minimal geodesic from \( B_- \) to \( B_+ \), parameterized by arc length, which we can assume starts at \( x_- \). The geodesic is orthogonal to \( B_- \) and hence to all orbits. Define \( x_+ = \gamma(L) \in B_+ \), \( x_0 = \gamma(\frac{L}{2}) \) and let \( K^\pm = G_{x_\pm} \) be the isotropy groups at \( x_\pm \) and \( H = G_{x_0} = G_{\gamma(t)}, 0 < t < L \), the principal isotropy group. Thus \( B_\pm = G \cdot x_\pm = G/K^\pm \) and \( G \cdot \gamma(t) = G/H \) for \( 0 < t < L \). For simplicity, we denote the tangent space of \( B_\pm \) at \( x_\pm \) by \( T_\pm \) and its normal space by \( T^\perp_\pm \).

By the slice theorem, we have the following description of the tubular neighborhoods \( D(B_-) = \pi^{-1}([0, L]) \) and \( D(B_+) = \pi^{-1}([\frac{L}{2}, L]) \) of the nonprincipal orbits:

\[
D(B_\pm) = G \times K^\pm \mathcal{D}^\pm,
\]

where \( \mathcal{D}^\pm \) are disks of radius \( \frac{L}{2} \) in \( T^\perp_\pm \). Here the action of \( K^\pm \) on \( G \times \mathcal{D}^\pm \) is given by \( k \cdot (g, \rho) = (gk^{-1}, kp) \) where \( k \) acts on \( \mathcal{D}^\pm \) via the slice representation, i.e., the restriction of the isotropy representation to \( T^\perp_\pm \). Hence we have the decomposition

\[
M = D(B_-) \cup E D(B_+),
\]