Chapter 11

Systems with Spontaneous Symmetry Breaking

Many physical systems, seemingly quite different from one another, turn out to be susceptible to investigation by certain topological methods. The phenomenon that unifies such physically different systems as liquid crystals, magnetism, and superfluid helium, is called spontaneous symmetry breaking. It underlies many contemporary concepts in the theory of elementary particles, the theory of phase transitions, and a number of problems of cosmology.

The essence of this phenomenon is revealed by a particular example taken from the theory of magnetism. Given a low enough temperature, a wide variety of crystals become magnetized in the absence of an external magnetic field. This phenomenon is called ferromagnetism and is explained by the existence of a special exchange interaction among the atoms of crystal lattices. The magnetization thus created is called spontaneous because it is formed without the application of an external field and is characterized by the magnetization vector $M$—the magnetic moment of a ferromagnet.

We will not delve more deeply into the labyrinth of the theory of the ferromagnet but will consider the classical model—the isotropic Heisenberg ferromagnet—as a spontaneous symmetry breaking effect. Consider a crystal lattice, at the vertices of which there are localized particles with half-integer spin, for example, electrons. For the sake of simplicity, we shall consider the lattice to be two-dimensional, although our conclusions are valid also for a three-dimensional lattice (Fig. 11.1). The interaction between electrons located in neighboring points of the lattice is defined by spin vectors $S(x)$. For the sake of definiteness we take the Hamiltonian (energy operator) $H$ to be

$$H = \sum_{x,x'} I(x - x')S(x)S(x').$$

(11.1)

The function $I(x) = 0$ if $x \neq a$, where $a$ is a base of the lattice $I(a) = \lambda$ ($\lambda < 0$). The quantity $I(x)$ is called the exchange integral, and the interaction itself is called a nearest-neighbor interaction, because the contribution in (11.1) involves only

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interactions between two adjacent points. The magnetization vector $M$ has the form

$$M = \sum_x S(x). \quad (11.2)$$

One can show that the Hamiltonian (11.1) does not vary with rotation of the magnetization vector $M$; that is, the energy does not depend on its direction. The state with the least energy—the ground state of the system—corresponds to the greatest value $\langle M^2 \rangle$. Remember that $I < 0$. In this state all spins $S$ are oriented identically along a certain fixed axis $n$ and the projection of the magnetization vector $M$ in the direction of the orientation of the spins $n$ has a definite value. From this it obviously follows that the ground state is not invariant relative to the full group of rotations of the vector $M$. The symmetry group must preserve the fixed direction of the vector $M$. In the given case, it corresponds to the group of rotations of a circle.

Now, at last, we can give the definition of spontaneous symmetry breaking. Systems in which the symmetry of the ground state does not correspond to the symmetry of the Hamiltonian are called systems with spontaneously broken symmetry. This is the accepted name for such systems, but it would be more correct to call them systems with hidden symmetry. In essence, the symmetry of the Hamiltonian is not broken, only hidden. In the ground state it is impossible to uncover the higher symmetry of the system.

Examples of similar symmetry-breaking are encountered in various problems of physics. It is known, for example, that nuclear forces are invariant relative to rotations; at the same time, the ground state of a nucleus with nonzero spin is not invariant relative to the group of rotations.

The effect of spontaneous symmetry breaking is one of the mechanisms that explain a broad range of phenomena—phase transitions in matter.

It is known that one and the same substance can be found in different states or phases, depending on external conditions (temperature, pressure, etc.). The transition from one phase to the other is called a phase transition. Phase transitions occur in a wide variety of materials. As an example, the transition of a metal from the normal