Chapter 9
Graph Polynomials and Their Applications I: The Tutte Polynomial

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Abstract  In this survey of graph polynomials, we emphasize the Tutte polynomial and a selection of closely related graph polynomials such as the chromatic, flow, reliability, and shelling polynomials. We explore some of the Tutte polynomial’s many properties and applications and we use the Tutte polynomial to showcase a variety of principles and techniques for graph polynomials in general. These include several ways in which a graph polynomial may be defined and methods for extracting combinatorial information and algebraic properties from a graph polynomial. We also use the Tutte polynomial to demonstrate how graph polynomials may be both specialized and generalized, and how they can encode information relevant to physical applications. We conclude with a brief discussion of computational complexity considerations.

Keywords  Tutte polynomial · Graph polynomial · Chromatic polynomial · Flow polynomial · Reliability polynomial · Shelling polynomial · Abelian sandpile model · Spanning tree · Beta invariant

MSC2000:  Primary 05-02; Secondary 05C15, 05A15, 05C99

9.1 Introduction

We begin our exploration of graph polynomials and their applications with the Tutte polynomial, a renowned tool for analyzing properties of graphs and networks. This two-variable graph polynomial, due to Tutte [101, 103, 104],

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has the important universal property that essentially any multiplicative graph invariant with a deletion/contraction reduction must be an evaluation of it. These deletion/contraction operations are natural reductions for many network models arising from a wide range of problems at the heart of computer science, engineering, optimization, physics, and biology.

In addition to surveying a selection of the Tutte polynomial’s many properties and applications, we use the Tutte polynomial to showcase a variety of principles and techniques for graph polynomials in general. These include several ways in which a graph polynomial may be defined and methods for extracting combinatorial information and algebraic properties from a graph polynomial. We also use the Tutte polynomial to demonstrate how graph polynomials may be both specialized and generalized, and how they can encode information relevant to physical applications.

We begin with the Tutte polynomial because it has a rich and well-developed theory, and thus it serves as an ideal model for exploring other graph polynomials in Chap. 10. Furthermore, because of the Tutte polynomial’s long history, extensive study, and its universality property, it is often a “point of contact” for research into other graph polynomials in that their study frequently includes exploring their relations to the Tutte polynomial. These interrelationships are a central theme of the Chap. 10.

In this chapter we give both recursive and generating function formulations of the Tutte polynomial, and state its universality in the form of a recipe theorem. We give a number of properties and combinatorial interpretations for various evaluations of the Tutte polynomial. We recover colorings, flows, orientations, network reliability, etc., and related polynomials as specializations of the Tutte polynomial. We discuss the coefficients, zeros, and derivatives of the Tutte polynomial, and conclude with a brief discussion of computational complexity.

9.2 Preliminary Notions

The graph terminology that we use is standard and generally follows Diestel [26]. Graphs may have loops and multiple edges. For a graph $G$ we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. An oriented graph, $\tilde{G}$, also called a digraph, has a direction assigned to each edge.

9.2.1 Basic Concepts

We first recall some of the notions of graph theory most used in this chapter. Two graphs $G_1$ and $G_2$ are isomorphic, denoted $G_1 \cong G_2$, if there exists a bijection $\phi : V(G_1) \rightarrow V(G_2)$ with $xy \in E(G_1)$ if and only if $\phi(x)\phi(y) \in E(G_2)$. We denote by $\kappa(G)$ the number of connected components of a graph $G$, and by $c(G)$ the number of nontrivial connected components, that is, the number of connected