Stochastic Insurance Models, Their Optimality and Stability

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Abstract: A class of discrete-time stochastic insurance models is investigated in the framework of cost approach, the aim being maximization of profit (or minimization of loss) during a finite or infinite time interval. Optimal and asymptotically optimal controls are established under the assumption that probability distributions of the claim process and premium inflow are known. Sensitivity analysis of the models with respect to cost parameter fluctuations and distribution perturbations is also provided.

Keywords and phrases: Stochastic insurance models, cost approach, optimal and asymptotically optimal policies

12.1 Introduction

Many centuries ago insurance companies were created for risk sharing and transferring. By paying a fixed money amount a risk averse policyholder obtains the guarantee of indemnification in case of an insured event (or risk) realization. According to legislation, fulfillment of liabilities to its clients is the primary task of any insurance company. Therefore by the beginning of the twentieth century the study of ruin probability was initiated by F. Lundberg and H. Cramér in the framework of collective risk theory. This subject was further developed by many researchers. Thus during the last century the reliability approach dominated in actuarial sciences.

Being a corporation, an insurance company has obligations to its shareholders as well. So, the secondary but very important task is to get profit and pay dividends. This problem has also attracted the attention of actuaries. The pioneering work of de Finetti (1957) was followed by many others, especially during the last decade. Investment policies and borrowing were treated as well; see, e.g., Schmidli (1994), Bulinskaya (2004, 2005). In particular, the company functioning after its “ruin” and reinstatement of solvability by shareholders, using their money to raise the company capital to some positive level, was treated, e.g., in Dickson and Waters (2004).

Investigation of insurance models in the framework of cost approach was initiated in Bulinskaya (2003). We concentrate below on a class of discrete-time insurance models
generalizing those introduced in Bulinskaya (2007a,b). Namely, we take into account that asset amounts to sell and money amounts to borrow cannot be infinite. It is reasonable to study discrete-time models since the financial balance is struck by the end of a calendar year and the duration of a reinsurance treaty is usually a year as well.

12.2 Model description

The aim of this research is to establish the optimal (and asymptotically optimal) policies of an insurance company minimizing its expected losses during a fixed planning horizon of \( n \) years, \( n \leq \infty \).

Suppose that \( \{\xi_i\}_{i \geq 1} \) is a sequence of i.i.d. nonnegative r.v.s with a finite mean and a density \( \varphi(s) > 0 \) for \( s \) belonging to some finite or infinite interval \([\kappa, \pi] \subset \mathbb{R}_+\). The corresponding distribution function \( F(t) = P(\xi_i \leq t) = \int_0^t \varphi(s) \, ds \). Here \( \xi_i \) is the excess of claims over premiums in the year \( i \).

Assume that by the end of a year the company can make one of the following decisions: I, sell some assets (immediately); II, borrow some money, the loan being available by the end of the next year; III, sell assets and borrow money.

Let \( x \) be the initial capital (if \( x < 0 \) its absolute value is the company debt) whereas \( c_1 \) is the loss incurred by selling the assets unit, \( c_2 \) is the interest rate while borrowing, \( r \) is the penalty for delay of a claim unit payment, and \( h \) is the inflation rate. For simplicity, we set the discount factor \( \alpha = 1 \). In contrast with the above-mentioned papers, here we take into account the following parameters: \( a_1 \), the assets amount available for sale, and \( a_2 \), the upper bound for a loan. Thus we have \( z_i \leq a_i, i = 1, 2, \) where \( z_1 \) is the amount sold and \( z_2 \) is the amount borrowed.

12.3 Optimal control

Denote by \( f_n(x) \) the minimal expected \( n \)-year costs. According to the Bellman optimality principle (see, e.g., Bellman, 1957), for \( n \geq 1 \),

\[
f_n(x) = \min_{0 \leq z_i \leq a_i, i = 1, 2} [c_1 z_1 + c_2 z_2 + L(x + z_1) + \mathbb{E} f_{n-1}(x + z_1 + z_2 - \xi_1)]
\]

where \( f_0(x) \equiv 0 \), \( L(v) = \mathbb{E}[h(v - \xi_1)^+ + r(\xi_1 - v)^+] \) and \( \mathbb{E} \) stands for expectation.

Putting \( v = x + z_1 \), \( u = v + z_2 \), and

\[
G_n(u, v) = (c_1 - c_2)v + c_2 u + L(v) + \int_0^\infty f_{n-1}(u - s)\varphi(s) \, ds
\]

one gets

\[
f_n(x) = -c_1 x + \min_{(u, v) \in D_x} G_n(u, v).
\]