Nonparametric Comparison of Several Sequential $k$-out-of-$n$ Systems

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Abstract: Sequential order statistics have been introduced to model sequential $k$-out-of-$n$ systems which, as an extension of $k$-out-of-$n$ systems, allow the failure of some components of the system to influence the remaining ones. Here, we consider nonparametric hypothesis testing for making the decision whether the baseline distributions of several sequential $k$-out-of-$n$ systems are equal. The asymptotic distribution of the test statistics are derived for the case of known model parameters and for the case where the model parameters of the systems are unknown.

Keywords and phrases: $k$-out-of-$n$ systems, sequential $k$-out-of-$n$ systems, counting processes, nonparametric $K$-sample tests

24.1 Introduction

An $n$ component system functioning as long as $k$ ($1 \leq k \leq n$) components work is called a $k$-out-of-$n$ system. Particular cases are parallel and series systems corresponding to $k = 1$ and $k = n$, respectively. The failure times $T_i$, $1 \leq i \leq n$, of the $n$ components are often assumed to be iid random variables; see, for example, Barlow and Proschan (1981), Meeker and Escobar (1998), and Navarro and Rychlik (2007) for the exchangeable case. The particular probabilistic model in which it is supposed that the failure times $T_i$, $1 \leq i \leq n$, of the $n$ components are iid random variables is hereinafter referred to as the common $k$-out-of-$n$ model. Implicit in this assumption is that the failure of any component of the system does not affect the remaining lifetime of the components that are still at work. In many situations, however, the assumption of the failure times being iid random variables may not be reasonable. For example, the failure of a high-voltage transmission line will increase the load put on the remaining high-voltage transmission lines, thus violating the iid assumption.

In this context, extended models have been proposed in the literature; see Kamps (1995a) for the sequential $k$-out-of-$n$ model as well as Hollander and Peña (1995) for an extension using a counting process approach. Both models are flexible in the sense that they allow for the distribution of the residual lifetime of the remaining components,
after the failure of some component, to change; i.e., the underlying failure rate of the remaining components is adjusted according to the number of preceding failures.

The sequential $k$-out-of-$n$ model is defined via random variables $X_1^*, \ldots, X_{n-k+1}^*$, which are called sequential order statistics. These random variables describe the failure times of a $k$-out-of-$n$ system. Thus, in the sequential $k$-out-of-$n$ model, the life length of a $k$-out-of-$n$ system is given by $X_{n-k+1}^*$. In the particular setting of sequential order statistics chosen here, the distribution of the random variables $X_1^*, \ldots, X_{n-k+1}^*$ is determined by a distribution function $F$, called the baseline distribution, and model parameters $\alpha_2, \ldots, \alpha_{n-k+1}$, which describe the adjustment of the failure rate of the remaining components according to the number of preceding failures. It is worth mentioning that, if we take $\alpha_2 = \cdots = \alpha_{n-k+1} = 1$, the random variables $X_1^*, \ldots, X_{n-k+1}^*$ are distributed as order statistics from a random sample of size $n$ with underlying distribution function $F$. Hence, the sequential $k$-out-of-$n$ model comprises the common $k$-out-of-$n$ model. For general theoretical properties and applications of sequential order statistics, one may refer to Cramer and Kamps (2001b, 2003), and Balakrishnan et al. (2008). Belzunce et al. (2003) consider conditions for certain aging properties of a vector of sequential order statistics (see also Hu and Zhuang, 2006). Comparison results for sequential order statistics can be found in Belzunce et al. (2005) and Zhuang and Hu (2007).

In this chapter, we discuss the problem of testing whether the underlying distribution functions of several sequential $k$-out-of-$n$ systems are equal. Formally, the problem is to decide if $K$ different sequential $k$-out-of-$n$ systems have the same underlying distribution function, i.e., to test

$$H_0 : F_1 = \cdots = F_K = F_0.$$  

There is a large literature on parametric methods for the same testing problem in the common $k$-out-of-$n$ system; one may refer to Kalbfleisch and Prentice (1980) and Lawless (2003). Parametric statistical inference for sequential order statistics in the $K$ sample case may be found in Cramer and Kamps (2001a). We concentrate here on nonparametric methods. The special case of $K = 2$ and a special weight function nonparametric method for the above problem are treated in Beutner (2008). Here we extend these results to arbitrary $K$ and a much wider class of weight functions.

In Section 24.2, we give a short description of sequential order statistics, derive the test statistics, and state some properties of counting processes based on sequential order statistics. Next, in Section 24.3, we derive the asymptotic distribution of the test statistic in the case where the model parameters $\alpha_2, \ldots, \alpha_{n-k+1}$ are known. Finally, in Section 24.4 the results are extended to the case where the model parameters $\alpha_2, \ldots, \alpha_{n-k+1}$ are unknown.

### 24.2 Preliminaries and derivation of the test statistics

#### 24.2.1 Sequential order statistics: Introduction and motivation

A definition of sequential order statistics with a view to the motivation given in the introduction can be found in Cramer and Kamps (1996). As shown in Cramer and Kamps (2003) they can also be defined as follows.