Chapter VIII.

Applications of the theory of the top.

Part A. Astronomical applications.

§1. The precession of the Earth’s axis, treated in association with an idea of Gauß.

Corresponding to the dominant position of astronomical applications in the older mathematical literature, the problem of the rotation of the Earth has been of conspicuous influence on the development of the theory of the top, as is proven, for example, by our borrowed nomenclature: regular precession, nutation, and line of nodes. We find the names of almost all the classical mathematicians associated with the history of this problem, beginning with Newton and continuing with Euler, d’Alembert, Laplace, Lagrange, and Poisson.

The theory of astronomical precession is very simple in the first approximation, and very complicated if an exhaustive treatment is attempted. The latter standpoint is taken in textbooks on astronomy;*) we must essentially adopt the former. To give the nonastronomical reader a glimpse of the laborious and admirable methods of astronomy, we present a few results of the more precise theory at the conclusion of this part of the chapter.

The difficulty increases enormously if we abandon the grounds of abstract dynamics and no longer regard the Earth as absolutely rigid. The debates that then occur are in no way closed at the present time. We will reserve this matter for the following part of the chapter, and first hold fast to the assumption of rigidity.

*) We refer in the following to Tisserand, Mécanique céleste, t. II, Chaps. 22–27. In §194, p. 442, Tisserand reports on the history of the problem and the contributions of the named classical mathematicians to his research.222
Our method is modeled after a procedure given by Gauß for the calculation of the secular perturbations of the planetary orbits. It has the advantage of great intuitiveness, and provides the individual components of the solution stepwise, according to the order of their importance. It appears not to have been applied to the present problem. Gauß himself introduced his method with the remark that “the secular variations of a planetary orbit due to the perturbation of another planet are the same, whether the perturbing planet actually describes its elliptical orbit according to Kepler’s law, or whether its mass is assumed to be distributed on the circumference of the ellipse in such a measure that equally large shares of the total mass are given to segments of the ellipse that are described in equally large times.”

We wish to appropriate this idea and broaden it: we will distribute not only the mass of the perturbing body along its orbit, but also, where it is later desirable (§2), the mass of the perturbed body, which we will then treat as a rigid ring; we will learn to find not only the secular perturbations, but also, on the basis of a different mass distribution, the periodic perturbations (§3).

As Gauß presented his method, it serves for the exact determination of the secular perturbations (at least those of the first order). In that we forgo the precision intended by Gauß, we will simplify, in that we first disregard the eccentricities of the orbits; that is, for us, the orbits of the Sun and the Moon. We therefore assume that these orbits are circular. The nonuniformity of the mass distribution in the quotation ellipse, is then eliminated, and gives way to a uniform distribution on the circumference of the circle.

The most important element of the rotational phenomena of the Earth is its precessional motion. The approximate kinematic relations for this motion are already known (page 50): the axis of the Earth forms an angle of 23 1/2° with respect to the normal to the ecliptic (more precisely, at the present time, 23° 27' 7'', which number, however, is slowly changing), and rotates about the named normal at this angle once in approximately 26,000 years. Together with the daily rotation of the Earth, this motion of the axis represents a regular precession in the previous

*) Determinatio attractionis etc., Ges. W. Bd. 3, p. 331 and 357. It is this same treatise that contains the single direct communication of Gauß on his theory of elliptic integrals.223