ON REPRESENTATIVES OF SUBSETS

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1. Let a set $S$ of $mn$ things be divided into $m$ classes of $n$ things each in two distinct ways, $(a)$ and $(b)$; so that there are $m$ $(a)$-classes and $m$ $(b)$-classes. Then it is always possible to find a set $R$ of $m$ things of $S$ which is at one and the same time a C.S.R. (= complete system of representatives) for the $(a)$-classes, and also a C.S.R. for the $(b)$-classes.

This remarkable result was originally obtained (in the form of a theorem about graphs) by D. König‡.

In the present note we are concerned with a slightly different problem, viz. with the problem of the existence of a C.D.R. (= complete system of distinct representatives) for a finite collection of (arbitrarily overlapping) subsets of any given set of things. The solution, Theorem 1, is very simple. From it may be deduced a general criterion, viz. Theorem 3, for the existence of a common C.S.R. for two distinct classifications of a given set; where it is not assumed, as in König's theorem, that all the classes have the same number of terms. König's theorem follows as an immediate corollary.

2. Given any set $S$ and any finite system of subsets of $S$:

\[(1) \quad T_1, T_2, \ldots, T_m;\]

we are concerned with the question of the existence of a complete set of distinct representatives for the system (1); for short, a C.D.R. of (1).

By this we mean a set of $m$ distinct elements of $S$:

\[(2) \quad a_1, a_2, \ldots, a_m;\]

such that

\[(3) \quad a_i \in T_i\]

($a_i$ belongs to $T_i$) for each $i = 1, 2, \ldots, m$. We may say, $a_i$ represents $T_i$.

It is not necessary that the sets $T_i$ shall be finite, nor that they should be distinct from one another. Accordingly, when we speak of a system of

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$k$ of the sets (1), it is understood that $k$ formally distinct sets are meant, not necessarily $k$ actually distinct sets.

It is obvious that, if a C.D.R. of (1) does exist, then any $k$ of the sets (1) must contain between them at least $k$ elements of $S$. For otherwise it would be impossible to find distinct representatives for those $k$ sets.

Our main result is to show that this obviously necessary condition is also sufficient. That is

**Theorem 1.** In order that a C.D.R. of (1) shall exist, it is sufficient that, for each $k = 1, 2, ..., m$, any selection of $k$ of the sets (1) shall contain between them at least $k$ elements of $S$.

If $A, B, ...$ are any subsets of $S$, then their *meet* (the set of all elements common to $A, B, ...$) will be written

$$A \cap B \cap ...$$

Their *join* (the set of all elements which lie in at least one of $A, B, ...$) will be written

$$A \cup B \cup ...$$

To prove Theorem 1, we need the following

**Lemma.** If (2) is any C.D.R. of (1), and if the meet of all the C.D.R. of (1) is the set $R = a_1, a_2, ..., a_\rho$ (possible 0, i.e. $R$ the null set), then the $\rho$ sets

$$T_1, T_2, ..., T_\rho$$

contain between them exactly $\rho$ elements, viz. the elements of $R$.

$R$ is, by definition, the set of all elements of $S$ which occur as representatives of some $T_i$ in every C.D.R. of (1).

To prove the lemma, let $R'$ be the set of all elements $a$ of $S$ with the following property: there exists a sequence of suffixes

$$i, j, k, ..., l', l$$

such that

$$a \in T_i,$$

$$a_j \in T_j,$$

$$a_j \in T_k,$$

$$...$$

$$a_l \in T_l,$$

and, further,

$$l \leq \rho.$$