ON REPRESENTATIVES OF SUBSETS

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1. Let a set $S$ of $mn$ things be divided into $m$ classes of $n$ things each in two distinct ways, $(a)$ and $(b)$; so that there are $m$ $(a)$-classes and $m$ $(b)$-classes. Then it is always possible to find a set $R$ of $m$ things of $S$ which is at one and the same time a C.S.R. (= complete system of representatives) for the $(a)$-classes, and also a C.S.R. for the $(b)$-classes.

This remarkable result was originally obtained (in the form of a theorem about graphs) by D. König.‡

In the present note we are concerned with a slightly different problem, viz. with the problem of the existence of a C.D.R. (= complete system of distinct representatives) for a finite collection of (arbitrarily overlapping) subsets of any given set of things. The solution, Theorem 1, is very simple. From it may be deduced a general criterion, viz. Theorem 3, for the existence of a common C.S.R. for two distinct classifications of a given set; where it is not assumed, as in König’s theorem, that all the classes have the same number of terms. König’s theorem follows as an immediate corollary.

2. Given any set $S$ and any finite system of subsets of $S$:

$$(1) \quad T_1, T_2, \ldots, T_m;$$

we are concerned with the question of the existence of a complete set of distinct representatives for the system (1); for short, a C.D.R. of (1).

By this we mean a set of $m$ distinct elements of $S$:

$$(2) \quad a_1, a_2, \ldots, a_m;$$

such that

$$(3) \quad a_i \in T_i$$

($a_i$ belongs to $T_i$) for each $i = 1, 2, \ldots, m$. We may say, $a_i$ represents $T_i$.

It is not necessary that the sets $T_i$ shall be finite, nor that they should be distinct from one another. Accordingly, when we speak of a system of

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58
of the sets (1), it is understood that \( k \) \textit{formally} distinct sets are meant, not necessarily \( k \) actually distinct sets.

It is obvious that, if a C.D.R. of (1) does exist, then any \( k \) of the sets (1) must contain between them at least \( k \) elements of \( S \). For otherwise it would be impossible to find distinct representatives for those \( k \) sets.

Our main result is to show that this obviously necessary condition is also sufficient. That is

\textbf{Theorem 1.} \textit{In order that a C.D.R. of (1) shall exist, it is sufficient that, for each \( k = 1, 2, \ldots, m, \) any selection of \( k \) of the sets (1) shall contain between them at least \( k \) elements of \( S \).}

If \( A, B, \ldots \) are any subsets of \( S \), then their \textit{meet} (the set of all elements common to \( A, B, \ldots \)) will be written

\[ A \wedge B \wedge \ldots. \]

Their \textit{join} (the set of all elements which lie in at least one of \( A, B, \ldots \)) will be written

\[ A \vee B \vee \ldots. \]

To prove Theorem 1, we need the following

\textbf{Lemma.} \textit{If (2) is any C.D.R. of (1), and if the meet of all the C.D.R. of (1) is the set \( R = a_1, a_2, \ldots, a_\rho \) (\( \rho \) can be 0, i.e. \( R \) the null set), then the \( \rho \) sets

\[ T_1, T_2, \ldots, T_\rho \]

contain between them exactly \( \rho \) elements, viz. the elements of \( R \).}

\( R \) is, by definition, the set of all elements of \( S \) which occur as representatives of some \( T_i \) in every C.D.R. of (1).

To prove the lemma, let \( R' \) be the set of all elements \( a \) of \( S \) with the following property: there exists a sequence of suffixes

\[ i, j, k, \ldots, l', l \]

such that

\[ a \in T_i, \]

\[ a_j \in T_j, \]

\[ a_j \in T_k, \]

\[ \ldots \]

\[ a_l \in T_l, \]

and, further,

\[ l \leq \rho. \]