Chapter 2
B-Spline Generated Frames

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Abstract B-splines are some of the most versatile functions in applied mathematics. The purpose of this chapter is to present the theory of frames in Hilbert spaces with a direct focus on B-spline generators.

2.1 Introduction

Frames provide a natural way of expanding functions in separable Hilbert spaces: They are more general than orthonormal bases and yield more flexibility. In this chapter we give a short presentation of general frame theory, as well as an introduction to frames in $L^2(\mathbb{R})$ having Gabor structure or wavelet structure. The main body of the chapter concerns explicit frame constructions based on B-splines.

The content can naturally be split into two parts: an introduction to frames in general Hilbert spaces, and concrete constructions in $L^2(\mathbb{R})$. The two parts are tied together by Section 2.7, where the B-splines are introduced.

We begin in Section 2.2 by considering the so-called Bessel condition: It is a technical condition implying that all the series expansions considered in this chapter converge unconditionally. Section 2.3 reminds the reader about bases, in particular, orthonormal bases, in Hilbert spaces; the important case of a Riesz basis is discussed in Section 2.4. Section 2.5 introduces frames and their central properties. Section 2.6 relates frames and Riesz bases; in particular, it turns out that all Riesz bases are frames.

Section 2.7 marks the beginning of the second part of the chapters, where we focus on concrete constructions. Most of these constructions are based on B-splines, so Section 2.7 gives a short presentation on their key properties. Section 2.8
deals with the basic properties of systems of functions formed by translates of a single function. Section 2.9 introduces Gabor systems and their frame properties, and Section 2.10 focuses on the case of tight frames. Section 2.11 states the main results from the theory for dual frames associated with a given Gabor frame; in Section 2.12 these results are used to construct explicitly given dual pairs of Gabor frames. Finally, Section 2.13 deals with wavelet frames generated by B-splines, in particular, the constructions obtained via the unitary extension principle due to Ron and Shen.

2.2 Bessel Sequences in Hilbert Spaces

The ultimate goal of the present chapter is to obtain series expansions in infinite-dimensional vector spaces. The purpose of the current section is to introduce a condition that ensures that the relevant infinite series actually converge.

Let $\mathcal{H}$ be a separable Hilbert space, with the inner product $\langle \cdot, \cdot \rangle$ chosen to be linear in the first entry. When speaking about a sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$, we mean an ordered set, i.e.,

$$\{f_k\}_{k=1}^{\infty} = \{f_1, f_2, \ldots\}.$$  

That we have chosen to index the sequence by the natural numbers is just for convenience: Soon, we will see that all results in this section (and all subsequent results based on the Bessel condition) hold with arbitrary countable index sets and the elements $f_k$ ordered in an arbitrary way.

We begin with a technical lemma.

**Lemma 2.1.** Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{H}$, and suppose that $\sum_{k=1}^{\infty} c_k f_k$ is convergent for all $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$. Then

$$T : \ell^2(\mathbb{N}) \to \mathcal{H}, \quad T \{c_k\}_{k=1}^{\infty} := \sum_{k=1}^{\infty} c_k f_k$$

defines a bounded linear operator. The adjoint operator is given by

$$T^* : \mathcal{H} \to \ell^2(\mathbb{N}), \quad T^* f = \{\langle f, f_k \rangle\}_{k=1}^{\infty}.$$  

Furthermore,

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq \|T\|^2 \|f\|^2, \quad \forall f \in \mathcal{H}.$$  

**Proof.** Consider the sequence of bounded linear operators

$$T_n : \ell^2(\mathbb{N}) \to \mathcal{H}, \quad T_n \{c_k\}_{k=1}^{n} := \sum_{k=1}^{n} c_k f_k.$$  

Clearly, $T_n \to T$ pointwise as $n \to \infty$, so by the principle of uniform boundedness, the map $T$ defines a bounded linear operator. In order to find the expression for $T^*$,