18
Hyperbolization of Fibrations

18.1. Compactness theorem for aperiodic homeomorphisms

Let \( S \) be a Riemann surface of finite hyperbolic type, \( \Gamma = \pi_1(S) \). We assume that \( \Gamma \) is embedded in \( PSL(2, \mathbb{R}) \) as a Fuchsian group so that \( S = \mathbb{H}^2 / \Gamma \). Recall that \( \bar{S} = (\mathbb{C} - \text{cl}(\mathbb{H}^2))/\Gamma \) has the same marked hyperbolic structure as \( S \) and the opposite orientation. These surfaces are identified via projection of the complex conjugation \( z \mapsto \bar{z} \) to a map \( S \rightarrow \bar{S} \). There is a continuous length function

\[
\ell : \mathcal{T}(S) \times \mathcal{ML}(S) \rightarrow \mathbb{R}
\]

that extends the length function for the closed geodesics in \( S \); see Section 11.16. Suppose that

\[
(\psi_n^+, \psi_n^-) \in \mathcal{T}(S) \times \mathcal{T}(\bar{S})
\]

is a sequence and \((\lambda^+, \lambda^-)\) is a pair of elements in \( \mathcal{ML}(S) \times \mathcal{ML}(\bar{S}) \) such that there is a number \( L < \infty \) such that for all \( n \geq 0 \),

\[
\ell(\psi_n^\pm, \lambda^{\pm}) \leq L.
\]

Consider the sequence \([\rho_n] \in \mathcal{T}(\Gamma)\) that corresponds to \((\psi_n^+, \psi_n^-)\) under the Bers isomorphism.

**Theorem 18.1** (Double Limit Theorem; W. Thurston, J.-P. Otal). Suppose that the pair of measured geodesic laminations \( \lambda^+, \lambda^- \) binds the surface \( S \). Then the sequence \([\rho_n]\) is subconvergent in the character variety \( \mathcal{R}(\Gamma, PSL(2, \mathbb{C})) \).
18. Hyperbolization of Fibrations

In the next section, we will give an outline of the proof; for details, the reader is referred to [Ota96] and [Thu87a].

Now we shall see how Theorem 15.16 can be deduced from the Double Limit Theorem. Suppose that $\tau : S \to S$ is an aperiodic homeomorphism. It has stable and unstable measured geodesic laminations $\lambda^+, \lambda^-$ that are binding the surface $S$. Recall that there is a constant $k > 1$ such that $C(\lambda^+) = k^{\pm 1} C(\lambda^-)$. The length of the laminations $\tau(\lambda^\pm)$ in the marked hyperbolic surface $\psi^\pm_n := \tau^\pm n(S)$ is the same as the length $L^\pm = \ell_S(\lambda^\pm)$ of $\lambda^\pm$ on $S$. Thus the length of $\lambda^\pm$ in $\psi^\pm_n$ is $L^\pm / k^n$, which tends to zero as $n \to \infty$. So we are in a situation when we can apply Theorem 18.1.

18.2. The Double Limit Theorem: An outline

We start the outline with the example where $\lambda^+ = \alpha, \lambda^- = \beta$ is a binding pair of simple closed geodesics in $S$. We will regard $\alpha, \beta$ as elements of the fundamental group $\Gamma$ of $S$. Suppose that the sequence of conjugacy classes of representations $[\rho_n] \in D_{\text{par}}(\Gamma, SL(2, \mathbb{C}))$ is not precompact in the character variety. Then (after rescaling by suitable factors $D_n$) the sequence $[\rho_n]$ subconverges to a minimal, small, relatively elliptic, and nontrivial action of the group $\Gamma$ on a tree $T$ (see Theorem 10.24). Theorem 4.84 claims that the translation lengths $\ell(\rho_n(\alpha))$ and $\ell(\rho_n(\beta))$ are uniformly bounded from above by $2L$. However, according to the construction of the tree $T$, this implies that both $\alpha$ and $\beta$ have fixed points on $T$:

$$
\ell_T(\alpha) = \lim_{n \to \infty} \ell(\rho_n(\alpha))/D_n = 0, \quad \ell_T(\beta) = \lim_{n \to \infty} \ell(\rho_n(\beta))/D_n = 0
$$

since $D_n \to \infty$. It follows from Theorem 11.34 that there is a (maximal elliptic) subsurface $B \subset S$ that contains homotopy classes of all loops whose representatives in $\pi_1(S) = \Gamma$ have fixed points in $T$. Our assumption that the pair $\alpha, \beta$ is binding implies that $S = B$, and therefore $\Gamma$ has a global fixed point in $T$. Contradiction.

Now we consider the general case. Suppose that the sequence $[\rho_n]$ is not precompact in the character variety. Recall that for each representation $\rho_n : \Gamma \to PSL(2, \mathbb{C})$, we have a number $D_n = D_{\rho_n}$ such that after rescaling $\mathbb{H}^3$ by $D_n^{-1}$, the sequence of representations $\rho_n$ is subconvergent to the action of the group $\Gamma$ on a tree $T$. We identified the space of projective classes of laminations $\mathcal{PML}(S)$ with a "unit sphere" $\Theta_S$ in $\mathcal{ML}(S)$ ($\Theta_S$ consists of normalized laminations $\mu/p(\mu)$; see Section 11.15). The subset $C(S)$ of simple closed geodesics in $S$ is embedded into $\Theta_S$ as $\gamma \mapsto \gamma/p(\gamma)$. Let $\Theta_S^0$ denote the image. According to Theorem 11.25, the subset $\Theta_S^0$ is dense in $\Theta_S$. The translation length $\ell_{\rho} : \Gamma \to \mathbb{R}$ defines a function

$$
\ell : \Theta_S^0 \times \mathcal{T}(\Gamma) \to \mathbb{R}
$$

by the formula

$$
\ell(\gamma/p(\gamma), \rho) = \ell(\rho(\gamma))/\|p(\gamma) \cdot D_{\rho}\|.
$$