March 16. A graph is said to be planar if it can be drawn in the plane with no crossing edges. A classic problem based on planarity is the "utility problem". Suppose there are three people—Jack, Jill, and Judy—living in separate houses, and also three utilities—water, gas, and electricity—each supplied by a different plant. We wish to connect each of the three houses to each of the three plants, but we don’t want any of the nine connections to cross each other. (Perhaps all of the utilities are supplied by cables or pipes buried just beneath the surface, and if two were to cross we might damage one conduit while installing the other. Oh well, nobody ever said this problem was practical, just that it was a classic.) We can easily make eight of the connections, but we run into trouble with the ninth, as shown below (left). Struggle as we may, we will be unable to make all nine connections. (We could cheat and run a conduit underneath one of the buildings, but this is not considered valid.) This graph is non-planar. The graph is shown on the right below; it is the complete bipartite graph on two sets of three vertices, meaning that it contains two sets of three vertices along with all edges joining a vertex in one set with one in the other set. This graph is usually denoted by $K_{3,3}$.

Planar graphs have many interesting properties. Let’s look at some planar graphs and see what we can observe. The graphs shown on the following page are planar “projections” of a cube and tetrahedron. (They are what we would see if we built the outlines of those figures (using wires for the edges, say) and then looked at them from points very close to the center of one face.) We’ll use $\mathcal{V}$ to represent the number of vertices in a graph, $\mathcal{E}$ the number of edges, and $\mathcal{F}$ the number of faces, where a face is a region that is bounded by edges of the graph and contains no other edges of the graph.
(For example, the graph of the cube has a square face in the center, four trapezoidal faces surrounding the square face, and one square "exterior" face, bounded by the four outer edges and consisting of the infinite region "outside" the graph.) The cube has $F = 6$, $V = 8$, and $E = 12$. The tetrahedron has $F = 4$, $V = 4$, and $E = 6$. In both cases, we observe that $V + F = E + 2$. This equation is often called Euler's formula, and is asserted by the following theorem: Any connected planar graph has $V + F = E + 2$. A planar graph that is not connected satisfies $V + F = E + 1 + C$, where $C$ is the number of connected components. (See page 184 for a formal definition of "connected components".)

![Graph](image)

We'll first prove the theorem for disconnected graphs by using the connected case (which we'll prove later). Consider a planar graph with $C \geq 2$. Let $V_i$, $F_i$, and $E_i$ be the number of vertices, faces, and edges, respectively, of the $i$th component. Then we know (from our assumption that the theorem is true for connected graphs) that

\[
V_1 + F_1 = E_1 + 2 \\
V_2 + F_2 = E_2 + 2 \\
\vdots \\
V_C + F_C = E_C + 2
\]

and hence

\[
(V_1 + V_2 + \cdots + V_C) + (F_1 + F_2 + \cdots + F_C) = (E_1 + E_2 + \cdots + E_C) + 2C.
\]

Clearly, $(V_1 + V_2 + \cdots + V_C) = V$ and $(E_1 + E_2 + \cdots + E_C) = E$. When we sum the $F_i$, however, we count each interior face exactly once, but we count the exterior face $C$ times, once per component. (The exterior face of a disconnected graph is an infinite region with two or more "holes" in it, one per component of the graph.) Since $F$ only counts this region once, we find that $(F_1 + F_2 + \cdots + F_C) = F + (C-1)$. 