Combinations and Permutations

January 5. In his first lecture, Pólya discussed in general terms what combinatorics is about: The study of counting various combinations or configurations. He started with a problem based on the mystical sign known, appropriately, as an "abracadabra".

The question is, how many different ways are there to spell out "abracadabra", always going from one letter to an adjacent letter? Due to the way some letters (especially C and D) are found only in certain rows, it turns out the only ways to spell "abracadabra" start with the topmost 'A' and zig-zag down to the bottommost 'A'. If we think of the letters as points, then any spelling of "abracadabra" specifies a sequence of points forming a crooked line from the top to the bottom. One such line is shown below.

You can also think of this problem in terms of a network of streets in a city where all blocks are the same size. Then the problem becomes one of computing how many ways there are of getting from the northern corner to the southern corner in the minimum number (10) of blocks. (That 10 is the minimum can be seen from the fact that each block, in addition to taking us either east or west, takes us
southward one-tenth the total southward distance between the two corners.)

It was decided empirically (i.e., by taking a vote) that there were more than 100 paths, but there was disagreement over whether there were more than 1000, so Pólya proceeded to approach the problem by more formal methods. He began by emphasising an important maxim which you should always consider when working on any problem: "If you cannot solve the proposed problem, solve first a suitable related problem." In this instance, the related problem is that of computing how many different paths there are from the northern corner to various other corners, still restricting ourselves to travelling only southeast and southwest. For starters, there is only one path to each of the corners on the northeast edge, namely the path consisting of travelling always southeast and never southwest. Similarly, there is only one path to each of the corners on the northwest edge. We note these values by writing them next to the corners involved.

Now what about the corner marked with a *? You could get there by going one block southeast followed by one block southwest, or by going first southwest and then southeast. Similarly, to get to the corner marked **, you could go southeast, then southwest twice, or