Chapter 10

Wolpert's Theorem

In this chapter we prove Wolpert's theorem that a generic compact Riemann surface is determined up to isometry by its length or eigenvalue spectrum. A new element in the proof given here is the observation that a finite part of the length spectrum determines the entire spectrum. With this and with some explicit computations based on trigonometry the proof becomes considerably shorter than the original one.

10.1 Introduction

A compact Riemann surface of genus \( g \geq 2 \) is determined up to isometry by the lengths of a finite system of closed geodesics. More precisely, let \( F \) be a base surface and let \( \mathcal{T}_g \) be the Teichmüller space of all marking equivalence classes of marked Riemann surfaces \((S, \varphi)\), where \( S \) is a compact Riemann surface of genus \( g \) and \( \varphi : F \to S \) is a marking homeomorphism (cf. Definitions 6.1.1 and 6.1.2). Consider a canonical curve system \( \Omega \) on \( F \) consisting of \( 9g - 9 \) simple closed curves as in Definition 6.1.6. For the present discussion it is only important to know that \( \Omega \) is some well defined finite ordered set of closed curves with certain useful properties. For any \( \beta \in \Omega \) and for any \( S = (S, \varphi) \) we let \( \beta(S) \) denote the simple closed geodesic in the free homotopy class of the curve \( \varphi \circ \beta \) on \( S \). In Corollary 6.2.8 we have seen that the marking equivalence class of \((S, \varphi)\) is uniquely determined by the ordered set of all lengths \( \ell \beta(S), \beta \in \Omega \). (The lengths \( \ell \beta(S) \) determine the Fenchel-Nielsen parameters.)

Hence, the position of an element in Teichmüller space is known if the lengths of sufficiently many closed geodesics are known, provided we know...
which length belongs to which closed geodesic.

In the length spectrum this information is not given. On the other hand, the length spectrum contains the lengths of all closed geodesics, and one may hope that this much larger amount of data may allow us to decode which length is attributed to which geodesic.

In [1] Gel'fand conjectured that this might be the case. As a first step towards the conjecture he proved that there exists no continuous deformation of a compact Riemann surface such that the spectrum remains fixed (cf. also Tanaka [1]). In [1] McKean showed that the number of pairwise non-isometric compact Riemann surfaces having the same spectrum is in fact finite. In [1] Wolpert succeeded to prove that the surfaces for which this number is different from 1 are located on a lower dimensional subvariety $\mathcal{V}_g$ of $\mathcal{F}_g$. This proved Gel'fand's conjecture in the generic case. At about the same time Marie France Vignéras showed that $\mathcal{V}_g$ is not empty. Hence Wolpert's result is optimal except that not much is known about $\mathcal{V}_g$ as yet.

Before we state Wolpert's theorem we review the definition of the length spectrum. In Definition 9.2.8 we considered the set $\mathcal{C}(M)$ of all oriented closed geodesics on the compact Riemann surface $M$. In the present chapter the orientation of a closed curve is irrelevant and we consider the non-oriented ones instead. We call parametrized closed geodesics

$$\gamma, \delta : S^1 = \mathbb{R}^1 / [t \mapsto t + 2\pi] \to M$$

(parametrized with constant speed) weakly equivalent if and only if there exists a constant $c$ such that either $\gamma(t) = \delta(t + c)$, $t \in S^1$ or $\gamma(t) = \delta(-t + c)$, $t \in S^1$. A non-oriented closed geodesic is the weak equivalence class of a closed parametrized geodesic. We could also introduce the non-oriented geodesics as the pairs of oppositely oriented oriented closed geodesics.

We recall that a closed geodesic is primitive if it is not the $m$-fold iterate of another closed geodesic for some $m \geq 2$ (Definition 1.6.4).

10.1.1 Definition. For any $S \in \mathcal{F}_g$ we let $\text{Lsp}(S)$ denote the sequence of the lengths of all non-oriented primitive closed geodesics on $S$, arranged in ascending order.

By Huber's theorem (Theorem 9.2.9) and by Remark 9.2.13, we have $\text{Lsp}(S) = \text{Lsp}(S')$ if and only if $S$ and $S'$ have the same spectrum of the Laplacian. In the present chapter we shall, however, not work with the eigenvalues.

In the following, "closed geodesic" always means "non-oriented closed geodesic". The "length spectrum of $S$" is, by definition, $\text{Lsp}(S)$. 