Chapter 7

The Spectrum of the Laplacian

This chapter gives a self-contained introduction into the Laplacian of compact Riemann surfaces. We prove the spectral theorem using the heat kernel which is given explicitly in Section 7.4 for the hyperbolic plane and in Section 7.5 for the compact quotients. As a tool we use the Abel transform which is introduced in Section 7.3. This transform will again show up in Chapter 9 in connection with Selberg's trace formula.

7.1 Introduction

We give a brief overview, with some history, of those topics in the spectral geometry of Riemann surfaces which will be covered in this book. Introductory texts for the Laplacian on Riemannian manifolds are Bérard [1], Berger-Gauduchon-Mazet [1] and Chavel [1]. A general overview of the Laplacian may be found in Simon-Wissner [1]. The articles by Elstrodt [2] and Venkov [1] give a detailed overview of the Laplacian on Riemann surfaces in connection with Selberg's trace formula. The article by Bérard [2] collects the developments in isospectrality up to 1989.

Originally, the Laplacian has been studied in mathematical physics in connection with the wave and the heat equations. See, for instance, Lagrange [1], Laplace [1], and Rayleigh [1] for the forerunners. One of the earliest spectral results which shows the interplay between the eigenvalues of the Laplacian and the geometry of the underlying domain is Weyl's asymptotic law [1, 2] of 1911:

\[
\lambda_k^{m/2}(M) \sim k \frac{c_m}{\text{vol } M}.
\]
Here $\lambda_k(M)$ is the $k$-th eigenvalue of the Laplacian on a compact domain $M \subset \mathbb{R}^n$ with respect to Dirichlet boundary conditions; $c_m$ is a dimension constant and $\sim$ denotes asymptotic equality as $k \to \infty$. Up to the year 1949 the spectrum of the Laplacian was studied mainly on domains in Euclidean space. In 1949 Minakshisundaram and Pleijel [1] published a fundamental article in which they gave a proof of the spectral theorem (Theorem 7.2.6) for an arbitrary compact Riemannian manifold $M$. Using the zeta function

$$Z(s) = \sum_{k=1}^{\infty} \frac{\lambda_k^s(M)}{s}, \quad s \in \mathbb{C}$$

(with real part $\Re s$ sufficiently large), they proved Weyl's law (7.1.1) for compact manifolds. They also showed that $Z(s)$ has an analytic continuation in the entire complex plane, thereby introducing methods from analytic number theory into Riemannian geometry. In the same year Maaß [1] introduced certain automorphic eigenfunctions of the Laplacian for Fuchsian groups of the first kind in connection with Dirichlet series and Siegel modular forms. The articles of Minakshisundaram-Pleijel and Maaß may be considered as the initiators of the “modern” spectral geometry of manifolds.

In 1954 Roelcke [1] proved general existence theorems for eigenfunctions and eigenpackets for Fuchsian groups of the first kind (this work was later continued in Elstrodt [1]). In the same year Huber [1] used the eigenfunction expansion of a new kind of Dirichlet series in order to study the asymptotic distribution of lattice points in the hyperbolic plane. At the same time Selberg [1, 2] undertook his investigations in harmonic analysis and found the celebrated Selberg trace formula (Section 9.5).

Huber [2] introduced a new geometric quantity, the *length spectrum* which is the sequence of the lengths of the closed geodesics listed in ascending order (cf. Definitions 9.2.8 and 10.1.1). He proved that for compact Riemann surfaces the length spectrum and the eigenvalue spectrum are equivalent geometric quantities. This result is also a consequence of Selberg’s trace formula, and we shall give a proof in Section 9.2 which lies somewhere between Huber’s and Selberg’s methods.

Since a finite set of the lengths (for instance those belonging to a canonical curve system) determine the Riemann surface up to isometry, it looks plausible that compact Riemann surfaces which are *isospectral*, i.e. which have the same spectrum, are isometric. This was conjectured by Gel’fand [1] in 1962. As a first step toward the conjecture, Gel’fand proved in [1] that continuous isospectral deformations cannot occur. On the other hand, Milnor [1] in 1964 gave examples of 16-dimensional isospectral non-isometric flat tori. Beyond this, little progress was made on behalf of the Laplacian on manifolds until the lecture notes of Berger-Gauduchon-Mazet [1] appeared in