Chapter 15
The Pseudo-Differential Operator Technique

The pseudo-differential operator theory emerged from the theory of singular integrals and Fourier analysis, having Kohn and Nirenberg as initiators. The theory was later extended and developed by Hörmander and has become an important tool in the theory of modern PDEs.

In this chapter we shall study the construction of the fundamental solution for heat operators using the symbolic calculus of pseudo-differential operators. After we provide the definition of pseudo-differential operators, we shall deal with the symbol of the product of pseudo-differential operators and provide an estimate for their multi-product. In this chapter we use pseudo-differential operators with both the usual symbols and Weyl symbols.

The main part is dedicated to the construction of the fundamental solution as a pseudo-differential operator with parameter $t$, for both nondegenerate and degenerate parabolic operators. In the case of the quadratic polynomial symbol of $(x, \xi)$, the exact symbol of the fundamental solution is obtained. These results are then applied to Grushin, sub-Laplacian, and Kolmogorov operators.

It is worth noting that the fundamental solution $E(t)$ obtained in this chapter is a smooth operator for any positive $t$ and $\int_0^c E(t) \, dt$ is a parametrix for any positive $c$.

This method has proved useful in proving the index theorem in its local version; see Gilkey [53] and Iwasaki [70, 71].

15.1 Basic Results of Pseudo-Differential Operators

We shall start with the definition of pseudo-differential operators following Hörmander [66] and Kumano-go [82], and then we shall provide the estimation of the symbol of multi-product. These computations will play an important role in the construction of the fundamental solution.
15.1.1 Definition of Pseudo-Differential Operators

In the following we shall define the set $S^m_{\rho,\delta}$ of symbols of order $m$ and type $(\rho, \delta)$, and organize it as a Fréchet space with respect to some semi-norms; see Hörmander [66] and Kumano-go [82]. We recall the well-known notations

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$

$$\partial^\alpha_x f(x) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f(x_1, \ldots, x_n),$$

for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

**Definition 15.1.1.** Let $m, \rho, \delta$ be real numbers such that $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. Then we denote by $S^m_{\rho,\delta}$ the set of all $C^\infty$-functions $p(x, \xi)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the following estimates for any multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$:

$$|\partial^\alpha_x \partial^\beta_\xi p(x, \xi)| \leq C_{\alpha,\beta} < \xi >^{-m+\rho|\alpha|+\delta|\beta|},$$

with a positive constant $C_{\alpha,\beta}$, where $< \xi > = (1 + |\xi|^2)^{1/2}$.

For simplicity we use the notation

$$p^{(\alpha)}_{(\beta)}(x, \xi) = \partial^\alpha_x \partial^\beta_\xi p(x, \xi).$$

For any symbol $p(x, \xi) \in S^m_{\rho,\delta}$, we define the semi-norms $|p|^{(m)}_\ell$, $\ell = 0, 1, 2, \ldots$, by

$$|p|^{(m)}_\ell = \max_{|\alpha|+|\beta| \leq \ell} \sup_{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n} \left\{ |p^{(\alpha)}_{(\beta)}(x, \xi)| < \xi >^{-m+\rho|\alpha|+\delta|\beta|} \right\}.$$  \hfill (15.1.1)

Then $S^m_{\rho,\delta}$ becomes a Fréchet space with respect to the semi-norms (15.1.1). The following notation will also be used:

$$S^{-\infty} = \bigcap_{-\infty < m < \infty} S^m_{1,0}.$$

In the following definition the prefix $Os$- will denote an *oscillatory integral*. More precisely,

$$Os - \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-iy \cdot \xi} f(y, \eta) dy d\eta = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-iy \cdot \xi} f(y, \eta) \chi_\varepsilon(y, \eta) dy d\eta,$$

where $\chi_\varepsilon(y, \eta) = \chi(\varepsilon y, \varepsilon \eta)$, with $\chi(y, \eta)$ a rapidly decreasing function such that $\chi(0, 0) = 1$. It is left as an exercise for the reader to show that the previous definition is independent on the cut function $\chi$. 
