Chapter 7
The Path Integral Approach

7.1 Introducing Path Integrals

In 1948 Feynman [42] provided an informal expression for the propagators of the famous Schrödinger equation, involving an integration over all the continuous paths with respect to a nonexistent infinite-dimensional Lebesgue measure. In fact, this is not really an integral, since there is no measure to give the integral. Since then, a large number of papers have tried to explain the precise mathematical meaning of the Feynman integral.

In this chapter we do not attempt this direction. We are concerned only with the immediate applications of the concept of a Feynman integral to obtaining heat kernels for differential operators.

In classical mechanics the particles follow trajectories provided by the Euler–Lagrange equations or Hamiltonian system. Given the initial conditions, standard theorems of ODE theory state the existence and uniqueness of the solution of the above systems of equations. This can be stated by saying that classical particles travel along deterministic trajectories.

Unlike the aforementioned case, in quantum mechanics particles travel on nondeterministic trajectories. This was inferred from the double-slit experiment and was theoretically stated by the Heisenberg principle of uncertainty.

Nondeterministic trajectories are better described by the notion of transition probability. Let $P(c \mid a)$ denote the probability that the particle is in state $c$ given that it starts at state $a$. An application of the conditional probability rule yields that the previous probability depends on all the intermediate states $b$:

$$P(c \mid a) = \sum_b P(c \mid b) P(b \mid a);$$

see Fig. 7.1b. In quantum mechanics the conditional probability $P(c \mid a)$ is replaced by the probability amplitude function $\varphi_{ca}$, in which case the aforementioned rule becomes

$$\varphi_{ca} = \sum_b \varphi_{cb} \varphi_{ba}.$$
Assume the particle starts at \( a \) at time zero and ends at \( b \) at time \( t \). Consider the following \( n \) intermediate states between \( a \) and \( c \):

\[
 a = b_0, \ b_1, \ldots, \ b_n, \ b_{n+1} = c,
\]

such that the particle is in state \( b_k \) at time \( t_k \), \( k = 0, \ldots, n \); see Fig. 7.2. The probability amplitude function between \( a \) and \( c \) can be written as

\[
\varphi_{ca} = \sum_{b_1, \ldots, b_n} \varphi_{cbn} \varphi_{b_{n-1}b_2b_1} \cdots \varphi_{b_2b_1} \varphi_{b_1a}.
\]

When the number of the intermediate states \( n \to \infty \), the linear function \( ab_1b_2 \cdots b_n c \) tends to a continuous path \( x(s) \) with \( x(0) = a \) and \( x(t) = c \). The “transition probability” from \( a \) to \( c \) at time \( t \) is obtained by summing over all the continuous paths between \( a \) and \( c \) parameterized by \([0, t]\):

\[
\varphi_{a,c;t} = \sum_{x(s)} \varphi_{x(s)}.
\]

A breakthrough idea came from Dirac, who indicated that \( \varphi_{x(s)} = e^{iS/\hbar} \) for \( t \) small, where \( S \) satisfies the Hamilton–Jacobi equation and \( \hbar \) denotes the Planck