The purpose of this chapter is to present a brief review of some basic set-theoretic concepts that will be needed in the sequel. By basic concepts we mean standard notation and terminology, and a few essential results that will be required in later chapters. We assume the reader is familiar with the notion of set and elements (or members, or points) of a set, as well as with the basic set operations. It is convenient to reserve certain symbols for certain sets, especially for the basic number systems. The set of all nonnegative integers will be denoted by $\mathbb{N}_0$, the set of all positive integers (i.e., the set of all natural numbers) by $\mathbb{N}$, and the set of all integers by $\mathbb{Z}$. The set of all rational numbers will be denoted by $\mathbb{Q}$, the set of all real numbers (or the real line) by $\mathbb{R}$, and the set of all complex numbers by $\mathbb{C}$.

1.1 Background

We shall also assume that the reader is familiar with the basic rules of elementary (classical) logic, but acquaintance with formal logic is not necessary. The foundations of mathematics will not be reviewed in this book. However, before starting our brief review on set-theoretic concepts, we shall introduce some preliminary notation, terminology, and logical principles as a background for our discourse.

If a predicate $P(\ )$ is meaningful for a subject $x$, then $P(x)$ (or simply $P$) will denote a proposition. The terms statement and assertion will be used as synonyms for proposition. A statement on statements is sometimes called a formula (or a secondary proposition). Statements may be true or false (not true). A tautology is a formula that is true regardless of the truth of the statements in it. A contradiction is a formula that is false regardless of the truth of the statements in it. The symbol $\Rightarrow$ denotes implies and the formula $P \Rightarrow Q$ (whose logical definition is “either $P$ is false or $Q$ is true”) means “the statement $P$ implies the statement $Q$”. That is, “if $P$ is true, then $Q$ is true”, or “$P$ is a sufficient condition for $Q$”. We shall also use the symbol $\not\Rightarrow$ for the denial of $\Rightarrow$, so that $\not\Rightarrow$ denotes does not imply and the formula $P \not\Rightarrow Q$
means “the statement $P$ does not imply the statement $Q$”. Accordingly, let $\neg P$ stand for the denial of $P$ (read: not $P$). If $P$ is a statement, then $\neg P$ is its contradiictory.

Let us first recall one of the basic rules of deduction called modus ponens: “if a statement $P$ is true and if $P$ implies $Q$, then the statement $Q$ is true” — “anything implied by a true statement is true”. Symbolically, $\{P \text{ true and } P \Rightarrow Q\} \implies \{Q \text{ true}\}$. A direct proof is essentially a chain of modus ponens. For instance, if $P$ is true, then the string of implications $P \Rightarrow Q \Rightarrow R$ ensures that $R$ is true. Indeed, if we can establish that $P$ holds, and that $P$ implies $Q$, then (modus ponens) $Q$ holds. Moreover, if we can also establish that $Q$ implies $R$, then (modus ponens again) $R$ holds.

However, modus ponens alone is not enough to ensure that such a reasoning may be extended to an arbitrary (endless) string of implications. In certain cases the Principle of Mathematical Induction provides an alternative reasoning. Let $\mathbb{N}$ be the set of all natural numbers. A set $S$ of natural numbers is called inductive if $n + 1$ is an element of $S$ whenever $n$ is. The Principle of Mathematical Induction states that “if 1 is an element of an inductive set $S$, then $S = \mathbb{N}$”. This leads to a second scheme of proof, called proof by induction. For instance, for each natural number $n$ let $P_n$ be a proposition. If $P_1$ holds true and if $P_n \Rightarrow P_{n+1}$ for each $n$, then $P_n$ holds true for every natural number $n$. The scheme of proof by induction works for $\mathbb{N}$ replaced with $\mathbb{N}_0$.

There is nothing magical about the number 1 as far as a proof by induction is concerned. All that is needed is a “beginning” and the notion of “induction”. Example: Let $i$ be an arbitrary integer and let $\mathbb{Z}_i$ be the set made up of all integers greater than or equal to $i$. For each integer $k$ in $\mathbb{Z}_i$ let $P_k$ be a proposition. If $P_i$ holds true and if $P_k \Rightarrow P_{k+1}$ for each $k$, then $P_k$ holds true for every integer $k$ in $\mathbb{Z}_i$ (particular cases: $\mathbb{Z}_0 = \mathbb{N}_0$ and $\mathbb{Z}_1 = \mathbb{N}$).

“If a statement leads to a contradiction, then this statement is false”. This is the rule of a proof by contradiction — reductio ad absurdum. It relies on the Principle of Contradiction, which states that “$P$ and $\neg P$ are impossible”. In other words, the Principle of Contradiction says that the formula “$P$ and $\neg P$” is a contradiction. But this alone does not ensure that any of $P$ or $\neg P$ must hold. The Law of the Excluded Middle (or Law of the Excluded Third — tertium non datur) does: “either $P$ or $\neg P$ holds”. That is, the Law of the Excluded Middle simply says that the formula “$P$ or $\neg P$” is a tautology. Therefore, the formula $Q \Rightarrow \neg P$ means “$P$ holds only if $Q$ holds”, or “$Q$ is a necessary condition for $P$”. If $P \Rightarrow Q$ and $Q \Rightarrow P$, then we write $P \Leftrightarrow Q$ which means “$P$ if and only if $Q$”, or “$P$ is a necessary and sufficient condition for $Q$”, or still “$P$ and $Q$ are equivalent” (and vice versa). Indeed, the formulas $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ are equivalent: $\{P \Rightarrow Q\} \iff \{\neg Q \Rightarrow \neg P\}$. This equivalence is the basic idea behind a contrapositive proof: “to verify that a proposition $P$ implies a proposition $Q$, prove, instead, that the denial of $Q$ implies the denial of $P$”.