Chapter 8
The Zak Transform(s)

Eugenio Hernández, Hrvoje Šikić, Guido L. Weiss, and Edward N. Wilson

Abstract The Zak transform has been used in engineering and applied mathematics for several years and many purposes. In this paper, we show how it can be used to obtain an exceedingly elementary proof of the Plancherel theorem and for developing many results in Harmonic Analysis in particularly direct and simple ways. Many publications state that it was introduced in the middle sixties. It is remarkable that only a small number of mathematicians know this and that many textbooks continue to give much harder and less transparent proofs of these facts. We cite a 1950 paper by I. Gelfand and a book by A. Weil, written in 1940 that indicate that in a general non-compact LCA setting the Fourier transform is an average of Zak transforms (which are really Fourier series expressions). We actually introduce versions of these transforms that show how naturally and simply one obtains these results.

8.1 Introduction

We introduce the operator Z that is often called the *Zak Transform*. Our definition is a bit different from the one that usually appears in the literature. We will discuss this difference and will also give a historical account that the reader may find particularly interesting. In order to do this, however, we need to present our treatment of the operator Z (and $\tilde{Z}$) which shows that the Fourier transform and its inverse are unitary as an immediate consequence of the basic properties of Fourier series.

The operator Z maps each $f \in L^2(\mathbf{R})$ into the function

$$(Zf)(x,\xi) = \sum_{k\in\mathbf{Z}} f(x+k)\mathrm{e}^{-2\pi \mathrm{i} k\xi} \equiv \varphi(x,\xi)\ , \tag{8.1}$$

H. Šikić (✉)
Department of Mathematics, University of Zagreb, Bijenička 30, HR-10000, Zagreb, Croatia
e-mail: hsikic@math.hr

J. Cohen and A.I. Zayed (eds.), *Wavelets and Multiscale Analysis: Theory and Applications*, Applied and Numerical Harmonic Analysis,
DOI 10.1007/978-0-8176-8095-4_8,

$x, \xi \in \mathbf{R}$. Let us explain the meaning of this equality. Since $f \in L^2(\mathbf{R})$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k \in \mathbf{Z}} |f(x+k)|^2 \mathrm{d}x = \int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x < \infty\,.$$

Thus,

$$\sum_{k \in \mathbf{Z}} |f(x+k)|^2 < \infty \tag{8.2}$$

for a.e. $x \in \mathbf{R}$. This means that for a.e. $x \in \mathbf{R}$ the series in (8.1) is the Fourier series of a function in $L^2([-\frac{1}{2}, \frac{1}{2}\rangle)$ (considered to be 1-periodic in ξ) we denote by $\varphi(x, \xi)$. Moreover, for a.e. $x \in \mathbf{R}$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\varphi(x, \xi)|^2 \mathrm{d}\xi = \sum_{k \in \mathbf{Z}} |f(x+k)|^2\,.$$

It follows, therefore, that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\varphi(x, \xi)|^2 \mathrm{d}\xi\, \mathrm{d}x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{k \in \mathbf{Z}} |f(x+k)|^2 \mathrm{d}x = \|f\|_2^2\,. \tag{8.3}$$

Z, therefore, maps $L^2(\mathbf{R})$ isometrically into a space of functions

$$\varphi \in L^2(\mathbf{T}^2) = L^2\left(\left[-\frac{1}{2}, \frac{1}{2}\right\rangle \times \left[-\frac{1}{2}, \frac{1}{2}\right\rangle\right).$$

Let us examine the space M of these images $\varphi(x, \xi) = (Zf)(x, \xi)$ of $L^2(\mathbf{R})$ under the transformation Z. We have seen that these images are functions of two real variables. Equality (8.3) asserts that the "norm" of φ involves only the variables $(x, \xi) \in \mathbf{T}^2$. The definition (8.1) indicates that $\varphi(x, \xi)$ should be 1-periodic in ξ. With respect to the variable x we have the easily established property

$$\varphi(x+j, \xi) = \mathrm{e}^{2\pi \mathrm{i} j \xi} \varphi(x, \xi) \tag{8.4}$$

for each $j \in \mathbf{Z}$. The property (8.4) tells us how $\varphi(x, \xi)$, for $x \in [-\frac{1}{2}, \frac{1}{2}\rangle$ and $\xi \in \mathbf{R}$, extends to all $x \in \mathbf{R}$. This shows that $|\varphi(x, \xi)|$ is 1-periodic in each of the variables $x, \xi \in \mathbf{R}$.

It is also easy to show that Z maps $L^2(\mathbf{R})$ onto $L^2(\mathbf{T}^2)$. Let $\varphi \in L^2(\mathbf{T}^2)$. Then $\int_{-\frac{1}{2}}^{\frac{1}{2}} |\varphi(x, \xi)|^2 \mathrm{d}\xi < \infty$ for a.e. $x \in [-\frac{1}{2}, \frac{1}{2}\rangle$. For each such x, φ, as a function of ξ, is a member of $L^2(\mathbf{T})$; thus, for a.e. $x \in [-\frac{1}{2}, \frac{1}{2}\rangle$, $\varphi(x, \xi)$ has a Fourier series

$$\varphi(x, \xi) \sim \sum_{k \in \mathbf{Z}} c_k(x) \mathrm{e}^{-2\pi \mathrm{i} k \xi}$$