Chapter 3
The Simplest Heteroclinics

Using the preliminary results of Chapter 2, the existence of heteroclinic solutions of (PDE) will be established in this section. To formulate the main result, set

\[ c_1 = c_1(v_0, w_0) \equiv \inf_{u \in \Gamma_1(v_0, w_0)} J_1(u). \]  

(3.1)

Theorem 3.2. If \( F \) satisfies \((F_1)\)–\((F_2)\) and \((*)_0\) holds,

1° There is a \( U_1 \in \Gamma_1 \) such that \( J_1(U_1) = c_1 \), i.e.,

\[ \mathcal{M}_1 = \mathcal{M}_1(v_0, w_0) \equiv \{ u \in \Gamma_1(v_0, w_0) \mid J_1(u) = c_1 \} \neq \emptyset; \]

2° Any \( U \in \mathcal{M}_1 \) satisfies

(a) \( U \) is a solution of \((PDE)\);

(b) \( \| U - v_0 \|_{C^2(T_i)} \to 0, i \to -\infty, \)

\( \| U - w_0 \|_{C^2(T_i)} \to 0, i \to \infty, \)

i.e., \( U \) is heteroclinic in \( x_1 \) from \( v_0 \) to \( w_0 \),

(c) \( v_0 < U < \tau_{-1}^1 U < w_0 \), i.e., \( U \) is strictly 1-monotone in \( x_1 \),

3° \( \mathcal{M}_1 \) is an ordered set.

Proof. Let \( (u_k) \subset \Gamma_1 \) be a minimizing sequence for \((3.1)\). Then there is an \( M > 0 \) such that for all \( k \in \mathbb{N} \),

\[ J_1(u_k) \leq M. \]  

(3.3)

Since \( J_1(u) = J_1(\tau_{-1}^1 u) \) for \( u \in \Gamma_1 \) via \((F_2)\), unless a normalization is imposed on \( (u_k) \), it may converge weakly to, e.g., \( v_0 \), yielding no useful information. Thus normalize \( u_k \) via

\[ \int_{T_i} u_k \, dx \leq \frac{1}{2} \int_{T_0} (v_0 + w_0) \, dx \leq \int_{T_0} u_k \, dx \]  

(3.4)
for all $i \in \mathbb{Z}$, $i < 0$, and for all $k \in \mathbb{N}$. This is possible by the definition of $\Gamma_1$.

Noting that $y = \Gamma_1(v_0, w_0)$ satisfies $(Y_1^i)$ of Proposition 2.50, by that result there is a $U_1 \in \hat{\Gamma}_1(v_0, w_0)$ such that $u_k \to U_1$ in $W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{T}^{n-1})$ along a subsequence that can be taken to be the entire sequence. By (3.4), for $0 > i \in \mathbb{Z}$,

$$\int_{T_i} U_1 \, dx \leq \frac{1}{2} \int_{T_0} (v_0 + w_0) \, dx \leq \int_{T_0} U_1 \, dx,$$

so $v_0 \neq U_1 \neq w_0$. By Proposition 2.8, (2.23), and the weak lower semicontinuity of $J_{1;p,q}$,

$$-K_1 \leq J_{1;p,q}(U_1) \leq M + 2K_1$$

for any $p \leq q$. Hence

$$-K_1 \leq J_1(U_1) \leq M + 2K_1. \tag{3.7}$$

To complete the proof, it will be shown that (A) $U_1$ is a solution of (PDE), as is any $U \in M_1$; (B) $U_1$ and any $U \in M_1$ satisfy $2^a(b)$ and $2^a(c)$; (C) $J_1(U_1) = c_1$, so $1^a$ holds, and lastly (D) $3^a$ is valid.

**Proof of (A).** For the first statement it suffices to verify $(Y_1^i)$ of Proposition 2.64 for $(u_k)$. Since $v_0 \leq u_k \leq w_0$, for $t_0 = t_0(\varphi)$ sufficiently small,

$$w_0 - 2 \leq v_0 - 1 \leq u_k + t\varphi \leq w_0 + 2.$$

Set $f_k = \max(u_k + t\varphi, w_0)$ and $g_k = \min(u_k + t\varphi, w_0)$. By Remark 2.77, it can be assumed that $f_k \in \Gamma_1(w_0)$. Hence by Theorem 2.72,

$$J_1(f_k) \geq 0. \tag{3.8}$$

Since $g_k \in \hat{\Gamma}_1(v_0 - 1, w_0)$, by (3.8),

$$J_1(g_k) \leq J_1(f_k) + J_1(g_k). \tag{3.9}$$

and as in (2.79)–(2.80),

$$J_1(f_k) + J_1(g_k) = J_1(u_k + t\varphi). \tag{3.10}$$

Set $\chi_k = \max(g_k, v_0)$ and $\psi_k = \min(g_k, v_0)$. Then $\chi_k \in \Gamma_1$ and $\psi_k \in \Gamma(v_0)$, so as in (3.8)–(3.10),

$$J_1(\chi_k) \leq J_1(\chi_k) + J_1(\psi_k) = J_1(g_k). \tag{3.11}$$

Combining (3.9)–(3.11) gives

$$c_1 \leq J_1(u_k) \equiv c_1 + \delta_k \leq J_1(\chi_k) + \delta_k \leq J_1(u_k + t\varphi) + \delta_k, \tag{3.12}$$

where $\delta_k \to 0$ as $k \to \infty$. Thus $(Y_1^i)$ holds and $U_1$ is a solution of (PDE). Next observe that if $U \in M_1$, the sequence $(\varphi_k)$, where $\varphi_k = U$ for all $k \in \mathbb{N}$, is a minimizing sequence for (3.1). Hence by what was just shown, $U$ is a solution of (PDE).