CHAPTER IV

Curves in \( \mathbb{R}^3 \)

4.1 Introduction

Though our main topic of concern is surfaces, prior to studying smooth surfaces we take a small detour through the study of smooth curves in \( \mathbb{R}^3 \) to develop some important tools. Our treatment of curves will be brief; more about curves, including such results such as the pretty Milnor–Fary Theorem, can be found in [M-P] or [DO1].

For the rest of the book we will be in the realm of differentiable functions. Section 4.2 reviews some basic facts concerning such functions, including the Inverse Function Theorem and some existence and uniqueness theorems for the solutions of ordinary differential equations, which play a foundational role for smooth surfaces.

4.2. Smooth Functions

We start with some assumptions about differentiable functions.

**Definition.** Let \( U \subset \mathbb{R}^n \) be a set, and let \( F: U \to \mathbb{R}^m \) be a map. We say \( F \) is smooth if

1. the set \( U \) is open in \( \mathbb{R}^n \); and
2. all partial derivatives of \( F \) of all orders exist and are continuous.

We can write \( F \) using coordinate functions as

\[
F(x) = \begin{pmatrix}
F_1(x) \\
\vdots \\
F_m(x)
\end{pmatrix},
\]

where \( x = \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right) \) and \( F_1, \ldots, F_m: U \to \mathbb{R} \) are smooth functions. The
Jacobian matrix of $F$ is the matrix of partial derivatives

$$ DF = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}. $$

The openness of the set $U$ in the above definition will often be unstated, but will be assumed nonetheless. The following definition is the smooth analog of the notion of homeomorphism.

**Definition.** Let $U, V \subset \mathbb{R}^n$ be open sets. A function $f: U \to V$ is a diffeomorphism if it is bijective, and if both $f$ and $f^{-1}$ are smooth.

If $f: U \to V$ is a diffeomorphism then the Jacobian matrix $Df$ is non-singular at each point in $U$ (see Exercise 4.2.2).

We now turn to the Inverse Function Theorem and differential equations; the reader should feel free to skip this material until it is needed in subsequent sections. The Inverse Function Theorem addresses the question of whether a smooth function $f: U \to \mathbb{R}^n$ has a smooth inverse (that is, whether it is a diffeomorphism). The one-dimensional case is simple. Let $f: J \to \mathbb{R}$ be a smooth function for some open interval $J$. If $f'(x_0) \neq 0$ for point $x_0 \in J$, then the function is either strictly increasing or strictly decreasing near $x_0$, and it follows that near $x_0$ the function has an inverse. Of course, having $f'(x_0) \neq 0$ does not imply that the whole function $f$ has an inverse, but only that the function restricted to some (possibly very small) open neighborhood of $x_0$ has an inverse. Since the graph of an inverse function is simply the reflection in the line $y = x$ of the original graph, we see that if $f'(x_0) \neq 0$ then the inverse function of $f$ restricted to a neighborhood of $f(x_0)$ will also be smooth. The Inverse Function Theorem is the higher-dimensional analog of what we have just discussed. The condition $f'(x_0) \neq 0$ is replaced by the condition that the Jacobian matrix has non-zero determinant at the given point.

**Theorem 4.2.1 (Inverse Function Theorem).** Let $U \subset \mathbb{R}^n$ be an open set and let $F: U \to \mathbb{R}^n$ be a smooth map. If $p \in U$ is a point such that $\det D F(p) \neq 0$, then there is an open set $W \subset U$ containing $p$ such that $F(W)$ is open in $\mathbb{R}^n$ and $F$ is a diffeomorphism from $W$ onto $F(W)$.

See [SK1, p. 34] and [BO, p. 42] for proofs, as well as other information concerning the Inverse Function Theorem. We will also need the following