The Degree Calculation

In this chapter, we will use the spectrum of a compact linear operator to establish an important fact about the Leray–Schauder degree. As in the previous chapter, $X$ is an infinite-dimensional Banach space.

The binomial theorem writes an expression of the form $(1 + x)^n$ as a polynomial in $x$ by using the operations of elementary algebra. These operations apply to $T \in L(X)$ as well, so the proof of the binomial theorem produces the same formula:

$$
(I - T)^n = \sum_{r=0}^{n} (-1)^r C^n_r T^r = I - \sum_{r=1}^{n} (-1)^{r+1} C^n_r T^r
$$

where

$$
C^n_r = \frac{n!}{r!(n-r)!}.
$$

Define a linear operator $S_n$ by setting

$$
S_n = \sum_{r=1}^{n} (-1)^{r+1} C^n_r T^r
$$

so that $(I - T)^n = I - S_n$. Since $K(X)$ is a linear subspace of $L(X)$ and compositions of compact operators are compact, if $T$ is compact so also is $S_n$. That means that everything we learned in Chapter 15 applies to the operator $(I - T)^n$.

For $T \in K(X)$, let $N_n$ denote the null space of $(I - T)^n = I - S_n$, then $N_n$ is finite dimensional by Lemma 15.7. Notice that $N_n$ is a subspace of $N_{n+1}$.

**Theorem 16.1.** For each $T \in K(X)$, there exists an integer $\nu$ such that $N_n \neq N_{n+1}$ for $n < \nu$ and $N_n = N_{n+1}$ for all $n \geq \nu$.

**Proof.** Suppose $N_n \neq N_{n+1}$ for all $n$. Since $N_n$ is closed in $N_{n+1}$, by Riesz' lemma for each $n$ there exists $x^*_{n+1} \in N_{n+1}$ with $\|x^*_{n+1}\| = 1$ and $\|x^*_{n+1} - v\| \geq \frac{1}{2}$ for all $v \in N_n$. This gives us a bounded sequence $\{x^*_{n+1}\}$ so, since $T$ is compact, $\{T x^*_{n+1}\}$ must contain a convergent sequence. But take any integers $r, s$ with $r < s$ and write
\[ T x_s^* - T x_r^* = x_s^* - (I - T)x_s^* + x_r^* - (I - T)x_r^*. \]

Now
\[
(I - T)^{-1}(I - T)x_s^* = (I - T)x_s^* = 0
\]
so \((I - T)x_s^* \in N_{s-1}\) and similarly \((I - T)x_r^* \in N_{r-1} \subseteq N_r \subseteq N_{s-1}\). Thus we can write \(T x_s^* - T x_r^* = x_s^* - x\) where
\[
x = (I - T)x_s^* + x_r^* - (I - T)x_r^* \in N_{s-1}.
\]
The property of \(x_s^*\) would then imply that \(I \|T x_s^* - T x_r^*\| \geq \frac{1}{2}\) and \(\{T x_n^*\}\) couldn’t contain a convergent sequence after all. The contradiction tells us that there must be some \(n\) for which \(N_n = N_{n+1}\). But then couldn’t we have \(N_n = N_{n+1}\) for some \(n\) and yet \(N_r \neq N_{r+1}\) for some \(r > n\)? It’s not hard to show that the answer is no, as follows. Let \(v\) be the minimum of the set of integers \(n\) such that \(N_n = N_{n+1}\) and suppose \(r > v\). For \(x \in N_r\), we have
\[
(I - T)^{v+1}(I - T)^{r-v-1}x = (I - T)^r x = 0
\]
so
\[
(I - T)^{r-v-1}x \in N_{v+1} = N_v.
\]
Consequently,
\[
0 = (I - T)^v(I - T)^{r-v-1}x = (I - T)^{r-1}x
\]
which tells us that \(N_r = N_{r-1}\).

For \(T \in K(X)\), we define \(R_n\) to be the range of \((I - T)^n\), which is a closed subspace of \(X\) by Theorem 15.9. Of course \(R_{n+1}\) is a subspace of \(R_n\). The \(R_n\) exhibit the same sort of stability behavior that we demonstrated for the \(N_n\) spaces in the previous result. The proof is very similar, so we’ll omit it.

**Theorem 16.2.** For each \(T \in K(X)\), there exists an integer \(\rho\) such that \(R_n \neq R_{n+1}\) for \(n < \rho\) and \(R_n = R_{n+1}\) for all \(n \geq \rho\).

**Lemma 16.3.** Let \(\rho\) be the integer defined by Theorem 16.2 and let \(m \geq 1\) be an integer, then \(N_m \cap R_\rho = 0\).

**Proof.** Let \(z \in N_m \cap R_\rho\). Choose \(n\) at least as large as both \(\rho\) and the integer \(\nu\) of Theorem 16.1, so in particular, \(N_n = N_{n+m}\) for all \(m\). Since \(R_n = R_\rho\), and \(z \in R_\rho\), there exists \(z_n \in X\) such that \((I - T)^n z_n = z\) which implies

\[
(I - T)^{n+m}z_n = (I - T)^m z = 0
\]
because \(z \in N_m\). Thus \(z_n\) is in \(N_{n+m} = N_n\) and therefore \(z = 0\).

**Theorem 16.4.** For \(T \in K(X)\), let \(\nu\) and \(\rho\) be the integers of Theorems 16.1 and 16.2, respectively, then \(\nu = \rho\).