Two-Dimensional Problems

We consider second-order partial differential equations which involve the scalar-valued dependent variable \( u = u(x, y) \). A simple example of the equations of this type is the Poisson's equation. We will present some examples of solving Poisson's and Laplace equations using the linear triangular and bilinear rectangular element.

6.1. Single Dependent Variable Problems

The finite element analysis of two-dimensional boundary value problems involves the following steps:

1. The boundary value problem is defined in a given domain \( \Omega \) by a second-order partial differential equation that is subject to prescribed boundary and initial values, and

2. The boundary \( \partial \Omega \) of the domain \( \Omega \) is a closed curve in most problems.

Thus, the finite elements for the domain \( \Omega \) are two-dimensional figures, such as triangles, rectangles, or quadrilaterals. A mesh of these elements covers the given domain, and the solution of the boundary value problem is approximated over this finite element mesh. Obviously, such a solution contains the discretization as well as approximation errors; the former error is because of the approximation of the domain, and the latter because of the approximation of the numerical solution.

We consider the general second-order equation

\[
\frac{\partial G_1}{\partial x} - \frac{\partial G_2}{\partial y} + c u - f = 0, \quad \text{in } \Omega, \tag{6.1}
\]
where \( c \) and \( f \) are known functions of \( x \) and \( y \), subject to the prescribed boundary conditions: \( u = \bar{u} \) on \( \Gamma_1 \), and \(- (G_1 n_x + G_2 n_y) = q_n \) on \( \Gamma_2 \), where \( \Gamma_1 \cup \Gamma_2 = \partial \Omega \) and \( \Gamma_1 \cap \Gamma_2 = \emptyset \), and

\[
G_1 \equiv a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y}, \quad G_2 \equiv a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y},
\]

with \( a_{ij} (i, j = 1, 2) \), are known functions of \( x \) and \( y \). Note that if \( a_{11} = a = a_{22} \), \( a_{12} = 0 = a_{21} \), and \( c = 0 \), then Eq (6.1) reduces to the Poisson's equation

\[
- \frac{\partial}{\partial x} \left( a \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left( a \frac{\partial u}{\partial y} \right) = f \quad \text{in} \ \Omega.
\]

A mesh of quadrilateral elements in the region \( \Omega \) is shown in Fig. 6.1. This mesh consists of different geometric figures of triangular, rectangular, or quadrilateral shapes. A typical element is denoted by \( \Omega^{(e)} \), and the discretization error is represented as the portions of the region (shaded in Fig. 6.1) between its boundary \( \Gamma \equiv \partial \Omega \) and the boundaries of the elements that lie toward the boundary \( \Gamma \).

\[\text{Fig. 6.1. Finite Elements.}\]

**6.1.1. Local Weak Formulation.** Let the essential boundary conditions be prescribed as \( u = \bar{u} \) on \( \Gamma_1 \), and the natural boundary condition as \(- (G_1 n_x + G_2 n_y) = q_n \) on \( \Gamma_2 \), as mentioned above. Let \( w \) denote a test function which vanishes on \( \Gamma_1 \). Then the weak variational form for an element \( \Omega^{(e)} \) is

\[
0 = \int_{\Omega^{(e)}} \left[ \frac{\partial w}{\partial x} \left( a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial w}{\partial y} \left( a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) + c w u - \omega f \right] dx \, dy + \int_{\Gamma^{(e)}} w q_n \, ds \equiv b(w, u) - I(w),
\]