More Two-Dimensional Problems

We will consider some two-dimensional steady-state boundary value problems from the areas of heat transfer problems (with and without convection), torsion, seepage, and fluid flows, and solve them by the finite element method.

7.1. Heat Transfer

The steady-state heat transfer in the \((x, y)\)-plane is governed by Eq (1.5), which is

\[-\frac{\partial}{\partial x} \left( k_x \frac{\partial T}{\partial x} \right) - \frac{\partial}{\partial y} \left( k_y \frac{\partial T}{\partial y} \right) = f(x, y), \quad (x, y) \in \Omega, \quad (7.1)\]

with the boundary \(\partial \Omega = \Gamma_1 \cup \Gamma_2\), where \(T\) denotes the temperature, \(f(x, y)\) the internal heat generation per unit volume, and \(k_x\) and \(k_y\) the thermal conductivities along the \(x\) and \(y\) axis, respectively. For heat conduction due to conduction or convection through the boundary \(\Gamma_2\), the natural boundary condition, in view of Newton's law of cooling, is

\[-k_x \frac{\partial T}{\partial x} n_x - k_y \frac{\partial T}{\partial y} n_y = \beta (T - T_\infty) + \dot{q}_n \quad \text{on} \ \Gamma_2, \quad (7.2)\]

where \(\beta\) denotes the convective heat transfer coefficient, \(T_\infty\) is the ambient temperature of the surrounding medium, \(\dot{q}_n\) the prescribed heat flow, and on \(\Gamma_1\) the temperature is prescribed as \(\bar{T}\). The weak form of Eqs (7.1) and (7.2) on an element
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\(\Omega(e)\) with the test function \(w\) is

\[
0 = \int_\Omega(e) \left( k_x \frac{\partial w}{\partial x} \frac{\partial T}{\partial x} + k_y \frac{\partial w}{\partial y} \frac{\partial T}{\partial y} - w f \right) \, dx \, dy
\]

\[
- \oint_{\Gamma(e)} w \left( k_x \frac{\partial T}{\partial x} n_x + k_y \frac{\partial T}{\partial y} n_y \right) \, ds
\]

\[
= \int_\Omega(e) \left( k_x \frac{\partial w}{\partial x} \frac{\partial T}{\partial x} + k_y \frac{\partial w}{\partial y} \frac{\partial T}{\partial y} - w f \right) \, dx \, dy
\]

\[
+ \oint_{\Gamma(e) \cap \Gamma_2} w \left[ q_n - \beta (T - T_\infty) \right] \, ds + \oint_{\Gamma(e) / \Gamma_2} w q_n \, ds
\]

\[
= b(w, T) - l(w).
\]

(7.3)

As mentioned in § 6.1.1, since Eq (7.3) holds for any test function \(w\), we choose this function in the form of \(n\) interpolation shape functions \(\phi_j(e)\), \(e = 1, \ldots, n\). Thus, to derive the finite element equation we approximate \(T\) by \(\sum_{j=1}^{n} T_j(e) \phi_j(e)(x, y)\), and replace \(w\) by \(\phi_i(e)\). Then Eq (7.3) yields

\[
\left( K^{(e)} + K_b^{(e)} \right) T^{(e)} = f^{(e)} + f_b^{(e)},
\]

(7.4)

where

\[
K_{ij}^{(e)} = \int_\Omega(e) \left( k_x \frac{\partial \phi_i(e)}{\partial x} \frac{\partial \phi_j(e)}{\partial x} + k_y \frac{\partial \phi_i(e)}{\partial y} \frac{\partial \phi_j(e)}{\partial y} \right) \, dx \, dy,
\]

\[
F_i^{(e)} = \int_\Omega(e) f \phi_i^{(e)} \, dx \, dy,
\]

(7.5)

\[
(K_b)_{ij}^{(e)} = \beta^{(e)} \oint_{\Gamma(e) \cap \Gamma_2} \phi_i^{(e)} \phi_j^{(e)} \, ds,
\]

\[
(f_b)_i^{(e)} = \beta^{(e)} \oint_{\Gamma(e) \cap \Gamma_2} \phi_i^{(e)} T_\infty \, ds.
\]

If we set \(\beta = 0\), we get the finite element model for heat conduction without convection. The boundary integrals for the triangular elements are discussed in Examples 6.2 and 6.3.

**Example 7.1.** Calculate the temperature distribution under steady-state heat conduction in an isotropic rectangular domain of length \(4a\) and width \(2a\) such that the boundaries \(x = 0\) and \(y = 0\) are insulated, the boundary \(x = 4a\) is kept at zero temperature, and the boundary \(y = 2a\) is maintained at a temperature