Chapter 11
On coalescence equations and related models

Philippe Laurençot and Stéphane Mischler

Summary. The Smoluchowski coalescence equation is a mean-field model for the growth of particles by successive mergers, and has been recently studied by deterministic and probabilistic methods. The present review article focuses on the deterministic approach and attempts to survey the currently available results on the questions of existence, uniqueness, mass conservation, gelation, and large time behaviour, while sketching the needed mathematical tools. When fragmentation is also taken into account and a detailed balance condition is assumed, recent techniques used to investigate the trend to equilibrium are outlined.

1 Introduction

Coalescence is a widespread phenomenon in nature and is one of the mechanisms by which particles (clusters) grow, the underlying process being successive mergers. In particular, coalescence phenomena are met and play an important role in various fields of physics (aerosol and raindrops formation, smoke, sprays, ...), chemistry (polymer, ...), astrophysics (formation of galaxies) and biology (hematology, animal grouping, ...) and take place at different scales [33]. At the level of particles, coalescence (or coagulation or aggregation) refers to mechanisms by which two (mother) particles encounter and merge into a single (daughter) particle. In the simple situation where each particle is fully identified by its mass (or size) $y \in Y$ (where $Y = \mathbb{N} \setminus \{0\}$ or $Y = \mathbb{R}_+ := (0, +\infty)$) and is denoted by $\{y\}$, the coalescence mechanism can be represented in a schematic way as

$$
\{y\} + \{y'\} \xrightarrow{a(y,y')} \{y + y'\},
$$

where $a$ stands for the probability (or rate) of occurrence of such an event. Let us observe right now that, during each coalescence event (1), the total mass is conserved, while the number of particles decreases and the mean size of the particles increases (from $(y + y')/2$ to $y + y'$).
From the modelling point of view, there are basically three levels of description of a system of a large number of particles undergoing coalescence events.

- **The microscopic level:** we consider a system of $N$ particles, $N \gg 1$, which evolves according to the coalescence mechanism (1), the coagulation events occurring in a random way. Such a description is mainly stochastic as was originally proposed by Smoluchowski [99, 100]. Among the stochastic models of coalescence, we mention the Markus–Lushnikov process [77, 82] which has been extensively studied recently [2, 55, 91]. Stochastic models of coalescence are currently an active field of research in probability theory and we refer to the survey by Aldous [2] and to [10, 27, 36, 37, 55, 56, 110] (and the references therein) for a more detailed account.

- **The mesoscopic level:** when we are not interested in the description of each identified particle in the system but rather in statistical properties of the system, a less accurate (mean-field) description is meaningful. We then introduce the statistical distribution $f(t, y) \geq 0$ of particles of mass $y \in Y$ at time $t \geq 0$ and mainly consider the time evolution of $f$. The most commonly used mean-field equation for $f$ is the celebrated Smoluchowski coagulation equation on which we will focus in this survey. We will discuss at length its main properties below, as well as some related models.

- **The macroscopic level:** the physically observable quantities are often averages of $f$ with respect to $y$ and a coarser description of the system can be reduced to the evolution of these quantities. However, the derivation of macroscopic equations for coalescence mechanisms is not yet clear and requires further investigations.

Still, there are links between these different levels and, in particular, the relationship between the microscopic and the mesoscopic levels is now well understood, and convergence proofs are available as well: convergence of the Marcus–Lushnikov process to the Smoluchowski equation [55, 91], Boltzmann–Grad limit of Smoluchowski’s description [60]. As for connections between the mesoscopic and the macroscopic levels, we are only aware of the recent work [39].

The aim of this survey is to present an overview of the mathematical analysis of coalescence equations and related models, and focus on the statistical description at the mesoscopic level. We point out some of the main mathematical problems and results with physical interest, as well as some mathematical tools and strategies that we think of for efficiency to investigate further these models. We also provide a (non-exhaustive) list of recent contributions and stress here that we mainly consider the deterministic approach to study coalescence models.

From now on, we thus restrict ourselves to the statistical description at the mesoscopic level and first consider the case where the distribution function $f = f(t, y)$ depends only on the time and mass variables $(t, y)$, that is, each particle is fully identified by its mass at the microscopic level. In order to understand the time evolution of $f$, the main issue is to figure out how exchange of mass takes place in the system. In fact, a central feature of the models considered herein is that mass “can be lost” during the time evolution. More precisely, the total mass of particles in the system $Y_1(t)$ at time $t$ given by