Chapter 6
Initial Value Problems for ODEs: Multistep Methods

We saw in Chap. 5 that (explicit) one-step methods are increasingly difficult to construct as one upgrades the order requirement. This is no longer true for multistep methods, where an increase in order is straightforward but comes with a price: a potential danger of instability. In addition, there are other complications such as the need for an initialization procedure and considerably more complicated procedures for changing the grid length. Yet, in terms of work involved, multistep methods are still among the most attractive methods. We discuss them along lines similar to one-step methods, beginning with a local description and examples and proceeding to the global description and problems of stiffness. By the very nature of multistep methods, the discussion of stability is now more extensive.

6.1 Local Description of Multistep Methods

6.1.1 Explicit and Implicit Methods

We consider as before the initial value problem for a first-order system of differential equations
\[
\frac{dy}{dx} = f(x, y), \quad a \leq x \leq b; \quad y(a) = y_0
\]  
(cf. Chap. 5, (5.14)–(5.16)). Our task is again to determine a vector-valued grid function \(u \in \Gamma_h[a, b]\) (cf. Chap. 5, Sect. 5.7) such that \(u_n \approx y(x_n)\) at the \(n\)th grid point \(x_n\).

A \(k\)-step method \((k > 1)\) obtains \(u_{n+k}\) in terms of \(k\) preceding approximations \(u_{n+k-1}, u_{n+k-2}, \ldots, u_n\). We call \(k\) the step number (or index) of the method.
We consider only linear \( k \)-step methods, which, in their most general form, but assuming a constant grid length \( h \), can be written as

\[
    u_{n+k} + \alpha_{k-1} u_{n+k-1} + \cdots + \alpha_0 u_n = h[\beta_k f_{n+k} + \beta_{k-1} f_{n+k-1} + \cdots + \beta_0 f_n], \quad n = 0, 1, 2, \ldots, N - k, \quad (6.2)
\]

where

\[
    x_r = a + rh, \quad f_r = f(x_r, u_r), \quad r = 0, 1, \ldots, N,
\]

and the \( \alpha \) and \( \beta \) are given (scalar) coefficients. The relation (6.2) is linear in the function values \( f_r \) (in contrast to Runge–Kutta methods); nevertheless, we are still dealing with a nonlinear difference equation for the grid function \( u \).

The definition (6.2) must be supplemented by a starting procedure for obtaining the approximations to \( y(x_s) \),

\[
    u_s = u_s(h), \quad s = 0, 1, \ldots, k - 1. \quad (6.4)
\]

These normally depend on the grid length \( h \), so may also the coefficients \( \alpha_s, \beta_s \) in (6.2). The method (6.2) is called explicit if \( \beta_k = 0 \) and implicit otherwise.

Implicit methods require the solution of a system of nonlinear equations,

\[
    u_{n+k} = h \beta_k f(x_{n+k}, u_{n+k}) + g_n, \quad (6.5)
\]

where

\[
    g_n = h \sum_{s=0}^{k-1} \beta_s f_{n+s} - \sum_{s=0}^{k-1} \alpha_s u_{n+s} \quad (6.6)
\]

is a known vector. Fortunately, the nonlinearity in (6.5) is rather weak and in fact disappears in the limit as \( h \to 0 \). This suggests the use of successive iteration on (6.5),

\[
    u_{n+k}^{[v]} = h \beta_k f(x_{n+k}, u_{n+k}^{[v-1]}) + g_n, \quad v = 1, 2, \ldots, (6.7)
\]

where \( u_{n+k}^{[0]} \) is a suitable initial approximation for \( u_{n+k} \). By a simple application of the contraction mapping principle (cf. Chap. 4, Sect. 4.9.1), one shows that (6.7) indeed converges as \( v \to \infty \), for arbitrary initial approximation, provided \( h \) is small enough.

**Theorem 6.1.1.** Suppose \( f \) satisfies a uniform Lipschitz condition on \([a, b] \times \mathbb{R}^d\) (cf. Chap. 5, Sect. 5.3),

\[
    \| f(x, y) - f(x, y^*) \| \leq L \| y - y^* \|, \quad x \in [a, b], \quad y, y^* \in \mathbb{R}^d, \quad (6.8)
\]

and assume that

\[
    \lambda := h |\beta_k| L < 1. \quad (6.9)
\]