Chapter 11
Solvable and Nilpotent Groups

Classes of Groups
By a class $\mathcal{K}$ of groups, we mean a subclass of the class of all groups with the following two properties:

1) $\mathcal{K}$ contains a trivial group
2) $\mathcal{K}$ is closed under isomorphism, that is,

$$G \in \mathcal{K} \quad \text{and} \quad H \cong G \implies H \in \mathcal{K}.$$

For example, the abelian groups form a class of groups. A group of class $\mathcal{K}$ is called a $\mathcal{K}$-group and $\mathcal{K}$-group $H$ that is a subgroup of a group $G$ is called a $\mathcal{K}$-subgroup of $G$. A class $\mathcal{K}$ is a trivial class if it contains only one-element groups.

Closure Properties
We will be interested in the following closure properties for a class $\mathcal{K}$ of groups:

1) (Subgroup)

$$G \in \mathcal{K}, \quad H \leq G \implies H \in \mathcal{K}$$

2) (Intersection and Cointersection) For $H, K \leq G$,

$$H, K \in \mathcal{K} \implies H \cap K \in \mathcal{K}$$

$$\frac{G}{H \cap K} \in \mathcal{K} \implies \frac{G}{H} \in \mathcal{K}$$

3) (Quotient and Extension) For $N \leq G$,

$$G \in \mathcal{K} \implies G/N \in \mathcal{K}$$

$$N, G/N \in \mathcal{K} \implies G \in \mathcal{K}$$
4) **(Seminormal Join, Normal Join and Cojoin)** For \( H, K \leq G \),

- \( H, K \in \mathcal{K} \), one normal in \( G \) \( \Rightarrow \) \( HK \in \mathcal{K} \)
- \( H, K \in \mathcal{K} \), both normal in \( G \) \( \Rightarrow \) \( HK \in \mathcal{K} \)
- \( \frac{G}{H}, \frac{G}{K} \in \mathcal{K} \) \( \Rightarrow \) \( \frac{G}{HK} \in \mathcal{K} \)

5) **(Direct product)**

\[ H, K \in \mathcal{K} \Rightarrow H \boxtimes K \in \mathcal{K} \]

These properties are not independent.

**Theorem 11.1** The following implications hold for a class \( \mathcal{K} \) of groups:

1) subgroup \( \Rightarrow \) intersection
2) quotient \( \Rightarrow \) cojoin
3) seminormal join \( \Rightarrow \) normal join \( \Rightarrow \) direct product
4) subgroup and direct product \( \Rightarrow \) cointersection

Thus, a class that is closed under

- subgroup, quotient, seminormal join, extension

is closed under all nine properties above.

**Proof.** Part 1) is clear. For part 2), we have

\[ \frac{G}{HK} \approx \frac{G}{H} / \frac{HK}{H} \in \mathcal{K} \]

For part 3), the direct product \( H \boxtimes K \) is the seminormal join of \( H \boxtimes \{1\} \) and \( \{1\} \boxtimes K \), each of which is in \( \mathcal{K} \) if \( H, K \in \mathcal{K} \). For part 4), if \( G/H, G/K \in \mathcal{K} \), then

\[ \frac{G}{H \cap K} \leftrightarrow \frac{G}{H} \boxtimes \frac{G}{K} \in \mathcal{K} \]

via the map \( \sigma : g(H \cap K) \mapsto (gH, gK) \).

The following definition will prove very convenient.

**Definition** Let \( \mathcal{K} \) be a class of groups.

1) A \( \mathcal{K} \)-series is a series whose factor groups belong to the class \( \mathcal{K} \).
2) A \( \mathcal{K}_s \)-group is a group that has a \( \mathcal{K} \)-series and a \( \mathcal{K}_n \)-group is a group that has a normal \( \mathcal{K} \)-series.
3) The \( \mathcal{K}_s \)-class is the class of all \( \mathcal{K}_s \)-groups and the \( \mathcal{K}_n \)-class is the class of all \( \mathcal{K}_n \)-groups.

Our main interest is in the \( \mathcal{K}_s \) and \( \mathcal{K}_n \) classes in which \( \mathcal{K} \) is either the class of cyclic groups or the class of abelian groups. However, we are also interested in a class of groups that is not a \( \mathcal{K}_s \) or \( \mathcal{K}_n \) class, namely, the nilpotent groups.