15. Lists of singularities

In this Section the beginning of the hierarchy of classes of singularities of holomorphic functions is described.

15.0 Preliminary remarks

1. Normal forms. A class of singularities is a subset of the space of germs (or jets) of functions that is invariant under the action of the group of diffeomorphisms of the source preserving the origin. Orbits are examples of classes of singularities. Two germs (or jets) are said to be equivalent if they belong to the same orbit.

Another example of a class is the so-called $\mu = \text{const}$ stratum. The multiplicity (Milnor number) $\mu$ of a critical point $0 \in \mathbb{C}^n$ of a function $f$ is the index of the singular point $0$ of the vector field $\text{grad} f$. The $\mu = \text{const} \text{ stratum for } f$ is defined as the connected component containing $f$ of the space of germs with fixed multiplicity $\mu$ at $0$.

To define a normal form, we regard the space of polynomials $M = \mathbb{C}[x_1, \ldots, x_n]$ as a subset of the space of germs of functions $f(x_1, \ldots, x_n)$ at $0$.

A normal form for a class $K$ of functions is given by a smooth map $\Phi: B \to M$ of a finite-dimensional linear space of parameters $B$ into the space of polynomials for which the following three conditions hold:

(1) $\Phi(B)$ intersects all the orbits of $K$;

(2) the inverse image in $B$ of each orbit is finite;

(3) the inverse image of the whole complement to $K$ is contained in some proper hypersurface in $B$.

A normal form is said to be polynomial (respectively, affine) if the mapping $\Phi$ is polynomial (respectively, linear and inhomogeneous). An affine normal form is called simple if $\Phi$ has the form

$$\Phi(b_1, \ldots, b_r) = \varphi_0 + b_1 x^{m_1} + \cdots + b_r x^{m_r},$$

where $\varphi_0$ is a fixed polynomial, the $b_i$ are numbers and the $x^{m_i}$ are monomials. (In the applications the polynomial $\varphi_0$ is usually “simple” itself, that is, a sum of a few monomials.)

The existence of a unique normal form (at least a polynomial one) for the
whole stratum $\mu = \text{const}$ is by no means obvious a priori. An astonishing conclusion from our calculations is the existence of such normal forms for all the singularities of our list (therefore, in particular, for all singularities with one and two moduli). The majority of our forms are simple; it is probable that all the singularities in Chapter 15 have simple normal forms. We do not know how extensive the class of functions is for which the stratum $\mu = \text{const}$ admits a simple (or at least polynomial) normal form (this question is natural for the stable equivalence classes). J. Wahl and V. A. Vasil'ev [178] have given an example of a $\mu = \text{const}$ stratum not admitting an affine normal form: to it belongs

$$f = x^2y^2(x + y)^2(x + 2y)^2 + x^9 - y^9.$$

2. Series of singularities. In the list the singularities are split into series, denoted by capital letters (we use light-face letters $A, \ldots, Z$ with various suffixes to denote the strata $\mu = \text{const}$ and bold-face letters $A, \ldots, Z$ with or without suffixes to denote classes of singularities that are unions of $\mu = \text{const}$ strata). Although the series undoubtedly exist, it is not at all clear what a series of singularities is.

For example, consider the series $A$ and $D$, which are formed by the orbits of the germs $A_k$: $f(x, y) = x^{k+1} + y^2$ and $D_k$: $f(x, y) = x^2y + y^{k-1}$. The classes $A_k$ and $D_k$ are adjacent to each other as follows*:

$$A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow A_4 \leftarrow \ldots$$

$$\uparrow \quad \uparrow$$

$$D_4 \leftarrow D_5 \leftarrow \ldots$$

It is clear that there are two series in this example, $A$ and $D$. However, what is the formal meaning of this assertion, and does it exceed the limits of the arbitrary names?

Thus, to define the series $A$ is to learn how to invert the arrows of the adjacencies, so that we go from $A_k$ to $A_{k+1}$, without deviating through $D_{k+1}$. In the given case this is not hard to do (the singularities of $A$ have a second differential of corank $\leq 1$). In more complicated cases we can also state a rule for inverting the arrows (in each case separately). As a result there arise series with one or several suffixes (for example, the three-suffix series $T_{k,l,m} = axyz + x^l + y^l + z^m$), and the functions of the series may depend on parameters.

As in the example of the series $A$ and $D$ above, so in all cases, after a series

* A class of singularities $L$ is adjacent to a class $K$ (notation: $K \leftarrow L$) if every function $f \in L$ can be deformed into a function of $K$ by an arbitrarily small perturbation.