17. Real, symmetric and boundary singularities

Three generalisations of the theory of critical points of functions are considered here. Tables are given of the simplest degeneracies in the real case, in the symmetric case and in the case of functions on manifolds with boundary.

17.1 Real functions

We shall consider smooth real functions with critical point 0 and critical value 0. The germs of two such real functions at 0 are said to be stably equivalent if they become equivalent (convertible into one another by \(R\)-equivalence, that is by a change of the independent variables) after the direct addition of nondegenerate quadratic forms. For example the germs of the functions \(f(x, y) = x^3 - y^2\) and \(g(x, y, z) = x^3 + y^2 + z^2\) are stably equivalent. Below is given the classification of the simple and the unimodal real germs up to stable equivalence.

*The simple germs.*

<table>
<thead>
<tr>
<th>(A_k^r, \ k \geq 1)</th>
<th>(D_k^r, \ k \geq 4)</th>
<th>(E_6)</th>
<th>(E_7)</th>
<th>(E_8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\pm x^{k+1})</td>
<td>(x^2 y \pm y^{k-1})</td>
<td>(x^3 \pm y^4)</td>
<td>(x^3 + xy^3)</td>
<td>(x^3 + y^5)</td>
</tr>
</tbody>
</table>

Remark: \(A_{2k}^r \sim A_{2k}, \ A_1^+ \sim A_1^-\); otherwise the germs shown are not equivalent. The start to the hierarchy of the degenerate singularities of real functions is the following:

![Diagram of singularities](image-url)
(the classes denoted by ... form a set of codimension 5 in the space of functions with critical point 0 and critical value 0).

*Unimodal germs* (according to V. V. Murav’ev and V. M. Zakalyukin).

<table>
<thead>
<tr>
<th>Notation</th>
<th>Normal form</th>
<th>Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parabolic:</td>
<td>$x^3 + ax^2z ± xz^2 + y^2z$</td>
<td>$a^2 ≠ 4$, if $+$</td>
</tr>
<tr>
<td>$P_8 = T_{3,3,3}$</td>
<td>$±x^4 + ax^2y^2 ± y^4$</td>
<td>$a^2 ≠ 4$, if $++$ or $−−$</td>
</tr>
<tr>
<td>$X_9 = T_{2,4,4}$</td>
<td>$x^3 + ax^2y^2 ± xy^4$</td>
<td>$a^2 ≠ 4$, if $+$</td>
</tr>
<tr>
<td>$J_{10} = T_{2,3,6}$</td>
<td>Hyperbolic of corank 2:</td>
<td>$a ≠ 0$, $k &gt; 0$</td>
</tr>
<tr>
<td>$J_{10+k} = T_{2,3,6+k}$</td>
<td>$x^3 ± x^2y^2 + ay^6 + k$</td>
<td>$a ≠ 0$, $k &gt; 0$</td>
</tr>
<tr>
<td>$X_{9+k} = T_{2,4,4+k}$</td>
<td>$±x^4 ± x^2y^2 + ay^4 + k$</td>
<td>$a ≠ 0$, $k &gt; 0$</td>
</tr>
<tr>
<td>$Y_{r,s} = T_{2,r,s}$</td>
<td>$±x^2y^2 ± x^r + ay^g$</td>
<td>$a ≠ 0$, $r, s &gt; 4$</td>
</tr>
<tr>
<td>$Y_r = T_{2,r,r}$</td>
<td>$±(x^2 + y^2)^2 + ax^r$</td>
<td>$a ≠ 0$, $r &gt; 4$</td>
</tr>
<tr>
<td>Hyperbolic of corank 3:</td>
<td>$P_{8+k} = T_{3,3,3+k}$</td>
<td>$a ≠ 0$, $k &gt; 0$</td>
</tr>
<tr>
<td>$P_{8+k} = T_{3,3,3+k}$</td>
<td>$x^3 ± x^2z + y^2z + az^4 + 3$</td>
<td>$a ≠ 0$, $k &gt; 0$</td>
</tr>
<tr>
<td>$R_{l,m} = T_{2,l,m}$</td>
<td>$x(x^2 + yz) ± y^l ± az^m$</td>
<td>$a ≠ 0$, $m ≥ l &gt; 4$</td>
</tr>
<tr>
<td>$R_m = T_{3,m,m}$</td>
<td>$x(±x^2 + y^2 + z^2) + ay^m$</td>
<td>$a ≠ 0$, $m &gt; 4$</td>
</tr>
<tr>
<td>$T_{p,q,r}$</td>
<td>$axyz ± x^p ± y^q ± z^r$</td>
<td>$a ≠ 0$, $p^{-1} + q^{-1} + r^{-1} &lt; 1$</td>
</tr>
<tr>
<td>$T_{p,q,r}$</td>
<td>$x^4 ± x^2y^2 ± ax^2y^3$,</td>
<td>$a ≠ 0$, $p^{-1} + 2m^{-1} &lt; 1$</td>
</tr>
<tr>
<td>$T_{p,q,r}$</td>
<td>$x^3 + y^7 ± z^2 ± axy^5$,</td>
<td>$W_{13}$</td>
</tr>
<tr>
<td>$E_{12}$</td>
<td>$x^3 + y^3 ± z^2 + ay^8$,</td>
<td>$Q_{10}$</td>
</tr>
<tr>
<td>$E_{13}$</td>
<td>$x^3 ± y^8 ± z^2 + axy^6$,</td>
<td>$Q_{11}$</td>
</tr>
<tr>
<td>$E_{14}$</td>
<td>$x^3y + y^5 ± z^2 ± axy^4$,</td>
<td>$Q_{12}$</td>
</tr>
<tr>
<td>$Z_{11}$</td>
<td>$x^2y + xy^4 ± z^2 + ax^2y^3$,</td>
<td>$S_{11}$</td>
</tr>
<tr>
<td>$Z_{12}$</td>
<td>$x^2y ± y^6 ± z^2 ± axy^5$,</td>
<td>$S_{12}$</td>
</tr>
<tr>
<td>$Z_{13}$</td>
<td>$±x^4 + y^5 ± z^2 ± ax^2y^3$,</td>
<td>$U_{12}$</td>
</tr>
<tr>
<td>$W_{12}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here $a$ is a real parameter.

The same theorems as in the complex case (p. 184) hold for the reduction to normal forms of generic singularities. The proofs are by the methods of Chapters 11–16.

Remark: Since the complex singularities have already been classified one can consider the real forms of each complex singularity. All the real-simple singularities are real forms of the complex-simple ones, while the real-unimodal ones are real forms of the complex-unimodal ones. However this fact is not obvious a priori and is only obtained by comparing the independently deduced complex and real classifications.