2. The classes $\Sigma^I$

Here singularities are classified according to the rank of the first differential of the map and the ranks of its restrictions to submanifolds of singularities.

2.1 The classification according to the degeneracy of the first differential

Let $f: M^m \to N^n$ be a smooth map, $f_x : T_x M^m \to T_{f(x)} N^n$ its derivative at $x$.

**Definition:** The point $x$ is said to be a point of class $\Sigma^i$ for $f$, if the dimension of the kernel of $f_x$ is equal to $i$. All the points of class $\Sigma^i$ for $f$ form a subset of $M$, called the set $\Sigma^i$ for $f$ and denoted by $\Sigma^i(f)$.

**Example:** For the map of Whitney pleat (Fig. 21)

![Diagram of Whitney pleat](https://example.com/whitney-pleat-diagram)

Fig. 21.

$M = \mathbb{R}^2$, $N = \mathbb{R}^2$, $f_1(x_1, x_2) = x_1^3 + x_1x_2$, $f_2(x_1, x_2) = x_2$

all the critical points are of class $\Sigma^1$, while the noncritical points are of class $\Sigma^0$.

**Remark:** In particular, the fold points and the pleat points are of the same class $\Sigma^1$.

The singularity of the map $w = z^2$ of the real plane at zero is of class $\Sigma^2$. By Whitney's theorem generic maps of two-dimensional manifolds do not have
singularities of class $\Sigma^2$. The question arises: what is the structure of the set $\Sigma^1(f)$ for a generic map $f:M^m \to N^n$? In particular, what is its dimension and when is it nonempty? To give an answer we require the

**Definition:** Let $A: \mathbb{R}^m \to \mathbb{R}^n$ be a linear operator of rank $r$. The differences $m - r$ and $n - r$ are called the coranks of $A$ at the source and target respectively.

**Remark:** The coranks are related to the dimension of the kernel $i$ by the obvious formulas: $m - r = i$, $n - r = n - m + i$.

**Theorem:** ("the corank product formula"). For a generic* map all the sets $\Sigma^1(f)$ are smooth submanifolds of the source space. The codimension of the manifold $\Sigma^1(f)$ is moreover equal to the product of the coranks:

$$\dim M - \dim \Sigma^1(f) = (m - r)(n - r)$$

(if this number is negative then the set is empty).

To understand where this formula comes from we consider first the corresponding problem in linear algebra.

### 2.2 The stratification of the space of linear operators

Consider the set of all linear operators $A: \mathbb{R}^m \to \mathbb{R}^n$. This is a linear space of finite dimension $mn$ (if a basis is chosen one can identify the operators with matrices of order $m \times n$). We shall denote this space by $L(m, n)$. The groups of linear changes of coordinates in the source space $GL(m)$ and in the target space $GL(n)$ act on the space of matrices $L(m, n)$ and give rise to a left-right action of the direct product of the two groups. Two matrices lie in the same orbit of this action if they are matrices of one and the same operator for different choices of bases in the source and target spaces.

The matrix of any operator $A$ can for a suitable choice of bases be put into the special form:

*The set of maps not satisfying the conclusion of this theorem is at worst a countable union of closed nowhere dense sets in the space of smooth maps; moreover if $M$ is compact then the set of "generic" maps in this context is open and everywhere dense.