Chapter 8

Convergence in Distribution

This chapter discusses the basic notions of convergence in distribution. Given a sequence of random variables, when do their distributions converge in a useful way to a limit?

In statisticians' language, given a random sample $X_1, \ldots, X_n$, the sample mean $\bar{X}_n$ is CAN; that is, consistent and asymptotically normal. This means that $\bar{X}$ has an approximately normal distribution as the sample size grows. What exactly does this mean?

8.1 Basic Definitions

Recall our notation that $df$ stands for distribution function. For the time being, we will understand this to correspond to a probability measure on $\mathbb{R}$.

Recall that $F$ is a df if

(i) $0 \leq F(x) \leq 1$;

(ii) $F$ is non-decreasing;

(iii) $F(x+) = F(x) \forall x \in \mathbb{R}$, where

$$F(x+) = \lim_{\epsilon \downarrow 0} F(x + \epsilon);$$

that is, $F$ is right continuous.
Also, remember the shorthand notation

\[ F(\infty) := \lim_{y \to \infty} F(y) \]
\[ F(-\infty) := \lim_{y \downarrow -\infty} F(y). \]

\( F \) is a probability distribution function if

\[ F(-\infty) = 0, \quad F(+\infty) = 1. \]

In this case, \( F \) is proper or non-defective.

If \( F(x) \) is a df, set

\[ C(F) = \{ x \in \mathbb{R} : F \text{ is continuous at } x \}. \]

A finite interval \( I \) with endpoints \( a < b \) is called an interval of continuity for \( F \) if both \( a, b \in C(F) \). We know that

\[ (C(F))^c = \{ x : F \text{ is discontinuous at } x \} \]

is at most countable, since

\[ \Lambda_n = \{ x : F([x]) = F(x) - F(x-) > \frac{1}{n} \} \]

has at most \( n \) elements (otherwise (i) is violated) and therefore

\[ (C(F))^c = \bigcup_n \Lambda_n \]

is at most countable.

For an interval \( I = (a, b] \), we write, as usual, \( F(I) = F(b) - F(a) \). If \( a, b \in C(F) \), then \( F((a, b)) = F(a, b] \).

**Lemma 8.1.1** A distribution function \( F(x) \) is determined on a dense set. Let \( D \) be dense in \( \mathbb{R} \). Suppose \( F_D(\cdot) \) is defined on \( D \) and satisfies the following:

(a) \( F_D(\cdot) \) is non-decreasing on \( D \).

(b) \( 0 \leq F_D(x) \leq 1 \), for all \( x \in D \).

(c) \( \lim_{x \in D, x \to +\infty} F_D(x) = 1 \), \( \lim_{x \in D, x \to -\infty} F_D(x) = 0 \).

Define for all \( x \in \mathbb{R} \)

\[ F(x) := \inf_{y \in D} F_D(y) = \lim_{y \uparrow x} F_D(y). \quad (8.1) \]

Then \( F \) is a right continuous probability df. Thus, any two right continuous df’s agreeing on a dense set will agree everywhere.