5.1 Pole and polar

We have seen that Poncelet used a way (in a sense, discovered by Brianchon) of pairing each point of the plane with a line, called the polar of the point, and each line in the plane with a point, called its pole. All that was needed for this was a conic. Any conic will do, and if a different conic is chosen the details of which point is paired with which line is changed, but nothing else. We also saw that the process of starting from a point (a pole) and producing a line (its polar) is the inverse of the process of starting from a line (a polar) and producing a point (its pole). We also saw that if three points lie on a line then their polars meet in a point and, conversely, if three lines meet in a point then their poles lie on a line. (See Figure 5.1.)

So Poncelet proclaimed a principle of duality, which said: given a conic, if you replace each line in a figure by its corresponding pole, and each point by its polar line, you obtain a new figure in which concurrent lines are replaced by collinear points and vice versa, and the original theorem becomes a theorem about the new figure on exchanging the words point and line, collinear and concurrent, throughout. (Such a pair of theorems are called dual theorems.) The simple act of dualising a theorem yields another theorem, this time about the dual figure.

Let us apply duality to Desargues’ theorem (see Figure 5.2). What do we get?
Answer: the converse! Notice that by cunningly choosing the names of the lines, the statement of the theorem and its converse involve nothing more than passing from upper- to lower-case letters.

Another amusing mathematical exercise is to dualise the construction of the fourth harmonic point (to construct what might be called a fourth harmonic line) and to check that it agrees with the construction of the fourth harmonic point.

Although Poncelet’s reasoning is correct, he generated a typically Parisian controversy. The idea of duality had been around for some time. One can, for example, dualise Pascal’s theorem, as a pupil of Monge, Charles Julien Brianchon, had already done in 1806 [27], to get this beautiful result about