Agner Krarup Erlang was born in 1878 in Lønborg, Denmark. His father was a schoolmaster. Between 1896 and 1901, the young Erlang studied mathematics, physics and chemistry at the University of Copenhagen. He then taught several years in high schools while keeping an interest in mathematics, especially probability theory. He met Jensen, chief engineer at the Copenhagen Telephone Company and an amateur mathematician, who convinced him in 1908 to join the new research laboratory of the company. Erlang started to publish articles on the applications of probability theory to the management of telephone calls. In 1917 he discovered a formula for waiting times, which was quickly used by telephone companies throughout the world. His articles, first published in Danish, were then translated in several other languages.

In 1929 Erlang became interested in the same problem of extinction that Bienaymé, Galton and Watson had studied before him for family names and that Fisher and Haldane had studied for mutant genes. Like his predecessors, he was not aware...
of all the works that had been published. Calling again \( p_k \) the probability for one individual to have \( k \) offspring, he noticed that the probability \( x_n \) of extinction within \( n \) generations satisfies

\[
x_n = p_0 + p_1 x_{n-1} + p_2 x_{n-1}^2 + \cdots = f(x_{n-1})
\]

with \( x_0 = 0 \). He noticed also that the overall extinction probability \( x_\infty \), which is the limit of \( x_n \) as \( n \to +\infty \), is a solution of the equation \( x_\infty = f(x_\infty) \). He realized that \( x = 1 \) was always a solution and that another solution existed between 0 and 1 when the average number of offspring \( R_0 = f'(1) \) is bigger than 1. But it seems that he could not figure out which of these two solutions was the right one. Like Galton, he submitted the problem in 1929 to a Danish mathematics journal, Matematisk Tidsskrift:

**Question 15.** When the probability that an individual has \( k \) children is \( p_k \), where \( p_0 + p_1 + p_2 + \cdots = 1 \), find the probability that his family dies out.

Unfortunately, Erlang died that same year 1929 at the age of 51. As a matter of fact, he died childless\(^1\).

A professor of actuarial mathematics at the University of Copenhagen, Johan Frederik Steffensen, took up Erlang’s question. He published in 1930 his solution in the same Danish journal: the probability of extinction \( x_\infty \) is always the smallest root of the equation \( x = f(x) \) in the closed interval \([0, 1]\), as Bienaymé and Haldane had already noticed. Steffensen’s proof is the one to be found in modern textbooks.

Indeed, we saw that the extinction probability \( x_\infty \) is a solution of \( x = f(x) \) in the closed interval \([0, 1]\). Let \( x^* \) be the smallest such solution. By definition, \( x^* \leq x_\infty \). Steffensen noticed first that \( x^* = f(x^*) \geq p_0 = x_1 \). Assume by induction that \( x^* \geq x_n \). Then \( x^* = f(x^*) \geq f(x_n) = x_{n+1} \) since the function \( f(x) \) is increasing. So \( x^* \geq x_n \) for all \( n \). Taking the limit, \( x^* \geq x_\infty \). So \( x_\infty = x^* \). Q.E.D.

Steffensen gave also a more formal explanation as to why \( x = 1 \) is the only root of \( x = f(x) \) when the mean number of offspring \( R_0 = f'(1) \) is smaller or equal to 1 (Fig. 18.2a) and why there is only one other root different from \( x = 1 \) in the case where \( R_0 > 1 \) (Fig. 18.2b). Notice that \( R_0 = f'(1) \) is the slope of the function \( f(x) \) at \( x = 1 \).

He noticed that for any root of \( x = f(x) \),

\[
1 - x = 1 - f(x) = 1 - p_0 - \sum_{k=1}^{+\infty} p_k x^k = \sum_{k=1}^{+\infty} p_k (1 - x^k).
\]

\(^1\) In his memory, the International Telephone Consultative Committee decided in 1946 to call “erlang” the unit of measure of the intensity of telephone traffic. “Erlang” is also the name given to a programming language by the company Ericsson.