Leonhard Euler was born in 1707 in Basel, Switzerland. His father was a Protestant minister. In 1720 Euler started studying at the university. He also received private mathematics lessons from Johann Bernoulli, one of the most famous mathematicians of the generation after Leibniz and Newton. He made friends with two of Johann Bernoulli’s sons: Nikolaus II and Daniel. In 1727 Euler joined Daniel at the newly created Academy of Sciences in Saint Petersburg. Apart from mathematics he was also interested in physics and many other scientific and technical subjects. In 1741 King Frederick II of Prussia invited him to become the director of the mathematics section of the Academy of Sciences in Berlin. Euler published a considerable number of articles and books on all aspects of mechanics (astronomy, elasticity, fluids, solids) and mathematics (number theory, algebra, infinite series, elementary functions, complex numbers, differential and integral calculus, differential and partial differential equations, optimization, geometry) but also on demography. He was the most prolific mathematician of his time.

Fig. 3.1 Euler (1707–1783)
In 1748 Euler published a treatise in Latin entitled *Introduction to Analysis of the Infinite*. He considered six examples in the chapter on exponentials and logarithms: one on the mathematical theory of musical scales, another on the repayment of a loan with interest, and four on population dynamics. In the latter Euler assumed that the population $P_n$ in year $n$ satisfies

$$P_{n+1} = (1 + x)P_n$$

for all integer $n$. The growth rate $x$ is a positive real number. Starting from an initial condition $P_0$, the population in year $n$ is given by

$$P_n = (1 + x)^nP_0.$$ 

This is called geometric or exponential growth. The first example asks:

If the population in a certain region increases annually by one thirtieth and at one time there were 100,000 inhabitants, we would like to know the population after 100 years.

The answer is $P_{100} = (1 + 1/30)^{100} \times 100,000 \simeq 2,654,874$. For this example Euler was inspired by the census of Berlin that took place in 1747 and which yielded an estimate of 107,224 for the population. His calculation shows that a population can increase more than tenfold within one century. This is precisely what had been observed at the time for the city of London.

One should note that computing $(1 + 1/30)^{100}$ is very easy with a modern pocket calculator. But in Euler’s time one had to use logarithms to avoid numerous multiplications by hand and get the result rapidly. One computes first the decimal logarithm (in base 10) of $P_{100}$. The fundamental property of the logarithm $\log(ab) = \log a + \log b$ shows that

$$\log P_{100} = 100 \log(31/30) + \log(100,000) = 100(\log 31 - \log 30) + 5.$$ 

Logarithms had been introduced in 1614 by the Scotsman John Napier. His friend Henry Briggs had published the first table of decimal logarithms in 1617. In 1628 the Dutch Adriaan Vlacq had completed Briggs’ work by publishing a table containing decimal logarithms of integers from 1 to 100,000 with ten-digit precision. This is the kind of table that Euler used to get $\log 30 \simeq 1.477121255$, $\log 31 \simeq 1.491361694$, and finally $\log P_{100} \simeq 6.4240439$. It remains to find the number $P_{100}$ whose logarithm is known. Since decimal logarithms of integers from 1 to 100,000 range from 0 to 5, one looks instead for the logarithm of $P_{100}/100$, which is 4.4240439. One can check in the table of logarithms that $\log 26,548 \simeq 4.424031809$ and $\log 26,549 \simeq 4.424048168$. Replacing the logarithmic function by a straight line between 26,548 and 26,549, Euler obtained that

$$\frac{P_{100}}{100} \simeq 26,548 + \frac{4.4240439 - 4.424031809}{4.424048168 - 4.424031809} \simeq 26,548.74.$$ 

So $P_{100} \simeq 2,654,874$. 