4 DYNAMICS OF FLUID FLOW

In this chapter the forces and energies involved in fluid flow will be considered. The conservation laws of mass, momentum and energy form the basis of solution for all problems in fluid dynamics.

4.1 CONSERVATION OF MOMENTUM

The law of conservation of momentum is embodied in Newton's second law of motion which states that the rate of change of momentum of a given mass is equal to the net force acting on the mass i.e.

$$\Sigma F = \frac{d}{dt} (mu) \quad (4.1)$$

This law is not a true conservation law in the sense that momentum is conserved under all conditions but if \( \Sigma F = 0 \) then momentum must be conserved.

4.2 EQUATIONS OF MOTION

The sum of the surface and body forces acting on a fluid element may be equated to the rate of change of momentum to obtain the equations of motion.

If shear or viscous forces are neglected we obtain the Euler equations of motion for an ideal fluid.

$$- \frac{1}{\rho} \frac{\partial p}{\partial x} + X = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \quad (4.2a)$$

$$- \frac{1}{\rho} \frac{\partial p}{\partial y} + Y = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \quad (4.2b)$$

$$- \frac{1}{\rho} \frac{\partial p}{\partial z} + Z = u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \quad (4.2c)$$

where \( X, Y \) and \( Z \) are the body forces per unit mass.

These three equations may be expressed in vector notation as

$$- \frac{1}{\rho} \text{grad } p + \mathbf{S} = \frac{\text{Du}}{\text{Dt}} \quad (4.3)$$

where \( \mathbf{S} = iX + jY + kZ \) and \( \mathbf{u} = iu + jv + kw \)

The Euler equations of motion are valid at any point in a flow field for the unsteady, three-dimensional, compressible flow of an ideal fluid.
If shear or viscous forces are included we obtain the Navier-Stokes equations of motion for a real fluid.

\[- \frac{1}{\rho} \text{grad} p + \mathbf{S} + \frac{1}{3} \frac{\mu}{\rho} \text{grad div} \mathbf{u} + \frac{\mu}{\rho} \nabla^2 \mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} \]  

(4.4)

The equations of motion are not amenable to complete solution and for integration over a finite region simplifying assumptions have to be made.

4.2.1 Irrotational Flow and Flow along a Streamline

Throughout the whole of an irrotational flow field or along a streamline in any ideal flow the Euler equations 4.2 reduce to a single integrable equation

\[- \frac{1}{\rho} \frac{dp}{dt} + X \frac{dx}{dt} + Y \frac{dy}{dt} + Z \frac{dz}{dt} = \frac{1}{2} \frac{d}{dt} \left( \frac{u^2}{2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial t} \frac{dt}{dt} \]  

(4.5)

4.2.2 Steady-state Euler Equation of Motion

For steady flow ($\partial p/\partial t = 0$) situations in which the sole body force is that due to gravity (weight), $Z = -g$ and $X = Y = 0$. Therefore

\[ \frac{dp}{\rho} + u \frac{du}{dt} + g \frac{dz}{dt} = 0 \]  

(4.6)

This is the steady-state Euler equation of motion which is strictly valid for the flow of an ideal fluid but it may be applied to real fluid flow if there are no significant velocity gradients due to eddies or solid surfaces. Integration is possible for incompressible flow where $\rho$ is constant or for compressible flow where $\rho$ is a known function of $p$ (e.g. isentropic flow $p/\rho^\gamma = \text{const}$).

4.3 THE BERNOULLI EQUATION

For constant density fluid, equation 4.6 may be integrated between two stations 1 and 2 in the flow to yield

\[ \frac{p_1}{\rho} + \frac{1}{2} u_1^2 + g z_1 = \frac{p_2}{\rho} + \frac{1}{2} u_2^2 + g z_2 = \text{constant} = e_0 \]  

(4.7a)

This is the Bernoulli equation which strictly applies to the steady frictionless flow of a constant density (incompressible) fluid, along a streamline, or the whole field of irrotational flow. Each of the terms has units of specific energy $N \text{ m/kg } (= m^2/s^2)$, therefore the Bernoulli equation is also an energy equation. It expresses conservation of 'mechanical' energy since each term is based on force quantities. Alternative forms of the Bernoulli equation are

\[ \frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2 = H_0 \]  

(4.7b)