CHAPTER XXII

HOMOGENEOUS FUNCTIONS AND EULER'S THEOREM

This too is an experience of the soul.
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1. Homogeneous functions

Often in a study of economics we come across the idea of "constant returns to scale". We may have, for example, that three men and ten acres will produce a certain amount of wheat, while six men and twenty acres will produce double that amount, nine men and thirty acres treble that amount and so on. This is just one simple example of linear homogeneous function. We now define these functions more precisely, and then consider a few of their properties.

Definition:
A function \( f(x_1, x_2, x_3, x_4, \ldots, x_n) \) is said to be homogeneous of degree \( n \) if

\[
f(tx_1, tx_2, tx_3, tx_4, \ldots, tx_n) = t^n f(x_1, x_2, x_3, x_4, \ldots, x_n)
\]

for all values of \( t \).

For example, if our function is

\[
f(x, y, z) = x^2 + y^2 + z^2
\]

then we have that

\[
f(tx, ty, tz) = (tx)^2 + (ty)^2 + (tz)^2
\]
\[= t^2 x^2 + t^2 y^2 + t^2 z^2
\]
\[= t^2(x^2 + y^2 + z^2)
\]
\[= t^2 f(x, y, z)
\]

and so \( x^2 + y^2 + z^2 \) is homogeneous of the second degree.
Similarly it can be shown that
\[ \frac{x + y}{x^4 - y^4} \] is homogeneous of degree \(-3\); \( f(tx, ty) = \frac{1}{t^3} f(x, y) \)
\[ x + y \] is homogeneous of degree \(1\); \( f(tx, ty) = tf(x, y) \)
and \( \frac{x}{y} \) is homogeneous of degree \(0\); \( f(tx, ty) = f(x, y) \), \( t^0 \) being unity.

**Exercise 22.1**

1. Consider whether the following functions are homogeneous, evaluating the degree of homogeneity where appropriate.
   - (i) \( x^3 + y^3 + 3z^3 \)
   - (ii) \( \sin x + \sin y \)
   - (iii) \( x^2 + xy + y^2 \)
   - (iv) \( x^2 + 2xy + y^2 \)
   - (v) \( x^2 + 2xy + 7y^2 \)
   - (vi) \( x + \frac{4x^3}{y} \)
   - (vii) \( x^2 + xy + y^2 + 1 \)
   - (viii) \( \frac{4x^3}{y} + 1 \)
   - (ix) \( \frac{x^2 + 4x^3}{y} \)
   - (x) \( \frac{x^2 + 4x^3}{y^2} + 1 \)

2. Euler’s theorem

Euler’s Theorem states that if \( f \) is a function of the variables \( x_1, x_2, x_3, \ldots, x_m \), and is homogeneous of degree \( n \) in these variables, then
\[ x_1 f_{x_1} + x_2 f_{x_2} + x_3 f_{x_3} + \ldots + x_m f_{x_m} = nf \]

We may illustrate this by referring to the examples just cited.

- (i) If \( f = x^2 + y^2 + z^2 \) (homogeneous of degree \(2\))
  
  then \( f_x = 2x, \ f_y = 2y \) and \( f_z = 2z \)

  Therefore, \( xf_x + yf_y + zf_z = 2x^2 + 2y^2 + 2z^2 = 2f \)

- (ii) If \( f = \frac{x + y}{x^4 - y^4} \) (homogeneous of degree \(-3\))

  we have by differentiation of the quotient that

  \[ f_x = \frac{(x^4 - y^4) - 4x^3(x + y)}{(x^4 - y^4)^2} \]

  and

  \[ f_y = \frac{(x^4 - y^4) + 4y^3(x + y)}{(x^4 - y^4)^2} \]