Field Problems—Analytical Solutions

The previous chapter has dealt with the main 'non-exact' methods of solving laplacian field problems, particularly of two dimensions. The application of numerical analysis to field problems and the advent of the digital computer has enabled complex field problems to be solved. However, these rely heavily on computer facilities, programming experience and particularly upon the suitability of the model used. Here one should note that it is not always desirable to model fields in terms of circuits and networks since this may well lead to a loss of physical insight into the nature of a particular field problem.

We now consider some of the analytical or 'exact' methods available for solving such problems.

9.1 Kelvin's method of images

This was introduced by Kelvin, in 1848, for solving electrostatic problems in which conducting boundary surfaces were present. Such a surface is shown in figure 9.1 in which we have region I containing two point charges $Q_1, Q_2$ and a
conducting equipotential boundary surface separating it from region II. The potential of the boundary is often chosen to be zero. It is possible to find a set of charges in region II, $Q'_1, Q'_2, Q'_3, \ldots$, so disposed that when the physical boundary is removed there is an exactly similar equipotential $V$ formed by the charge system, and the field in region I remains unaltered.

The utility of the method lies entirely in the simplicity of the boundary surface and the ease with which the values and positions of the hypothetical charges $Q'_1, Q'_2, \ldots$ can be located.

There are some points of similarity with the images of light sources in mirrors but the duality often breaks down particularly when iron boundaries and positive and negative currents are considered. A few examples will be discussed to illustrate the method.

9.1.1 Point charge and conducting plane

This is the classic problem and a point charge in the vicinity of a conducting plane is shown in figure 9.2a.

![Figure 9.2 Point charge and conducting plane.](image)

The conducting plane and its induced electric surface charge produce the same field in region I as a charge $-Q$ placed in the mirror-image position and the boundary plane removed. The field is illustrated in figure 9.2b and its strength at any point $P$ is given by

$$E = \frac{Q}{4\pi\varepsilon_0 r_1^2} a_{r_1} - \frac{Q}{4\pi\varepsilon_0 r_2^2} a_{r_2}$$

In particular this enables us to find the electric displacement $D$ at the boundary plane and the induced charge density there.

The same image system applies for a line charge $\rho_L$ which is parallel to an infinite conducting plane; the field equation becomes

$$E = \frac{+\rho_L}{2\pi\varepsilon_0 r_1} a_{r_1} - \frac{\rho_L}{2\pi\varepsilon_0 r_2} a_{r_2}$$

Expressions for the potential $V$ can be obtained from these equations since $E = -\nabla V$. 