7.1 Introduction

Non-linear theories of the firm stem very much from the work of the neoclassicists. Its very non-linearity is both a gain and a limitation. It is a gain in the sense that it introduces a greater realism at the firm level, but it is a limitation in the sense that it is almost impossible to generalise to the economy level. The non-linear approach rests almost wholly on the differential calculus as its tool of analysis. This is not hard to understand when one realises that the margin is no more than the first derivative of a function. We have already seen in the theory of consumer behaviour that the margin was central in discussions of marginal utility and the marginal rate of substitution. In the theory of the firm also the margin plays a central role in marginal product, marginal rate of technical substitution, and marginal cost.

The central character in the non-linear theory of production is the production function, which we shall discuss in the next section. It is worthy of attention in its own right because so much of the literature is either directly or indirectly concerned with it. For example, the cost functions follow on from and are not independent of the form of the production function.

7.2 The Production Function

A production function denotes the maximum output that can be obtained from a set of inputs. It is a function in the mathematical sense because it is assumed that to each set of inputs there is a unique maximum output. Hence, a production function is a mapping from the input space, X, into the output space, which we shall denote \( \bar{Q} \). Let \( x = (x_1, \ldots, x_s) \) denote a vector of inputs \( x \geq 0, x \in X \), and let \( q \in \bar{Q} \) denote the quantity of output. We shall assume there is a single output. For any \( x \in X \) there is a unique \( q \in \bar{Q} \) which is the maximum.
Definition 7.1

If $X$ is the input space and $Q$ the output space, a production function is a mapping of the form:

$$ f: X \rightarrow Q \quad \text{or} \quad q = f(x_1, \ldots, x_s). $$

In the present section we shall be concerned with the properties of $f$ and the various forms it has taken in the literature. Two commonly used production functions, chosen for their mathematical properties and possibility of being empirically estimated, are:

(a) The Cobb–Douglas production function, which for the two-input case takes the form

$$ q = a_0 x_1^{a_1} x_2^{a_2} \quad a_0, a_1, a_2 > 0 $$

or more generally,

$$ q = a_0 \prod_{i=1}^{s} x_i^{a_i} \quad a_i > 0 \forall i. $$

(b) The Constant Elasticity of Substitution production function, which for the two-input case takes the form

$$ q = a_0 [a_1 x_1^\rho + a_2 x_2^\rho]^{-\nu/\rho} \quad a_0, a_1, a_2 > 0, \nu > 0, \rho \geq -1 $$

or more generally,

$$ q = a_0 \left[ \sum_{i=1}^{s} a_i x_i^\rho \right]^{-\nu/\rho} \quad a_i > 0 \forall i, \nu > 0, \rho \geq -1. $$

In this section we shall consider only the continuous forms of $f$.

What makes $f$ continuous? If the inputs are continuously divisible then the input space, $X$, is connected. Even if this requirement is met, however, we cannot establish the continuity of the production function: we can only assume it to be continuous, and then derive the implications of such an assumption.

Assumption The production function $f: X \rightarrow Q$ is continuous.

Even if $f$ is continuous it may not be differentiable at every point $x \in X$. If $f$ is differentiable at least up to the second order then by assumption $f_i = \partial q/\partial x_i$ and $f_{ij} = \partial^2 q/\partial x_i \partial x_j$ exists for all $i, j = 1, \ldots, s$; and are themselves continuous. Let $q_x$ denote the column vector of all first derivatives, and $Q$ the matrix of all second-order derivatives, i.e.

$$ q_x = \frac{\partial q}{\partial x} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_s \end{bmatrix}, \quad Q = \frac{\partial q_x}{\partial x'} = \begin{bmatrix} f_{11} & f_{12} & \ldots & f_{1s} \\ f_{21} & f_{22} & \cdots & f_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ f_{s1} & f_{s2} & \cdots & f_{ss} \end{bmatrix}. $$