3 Exact Arbitrage: A Capital Asset Pricing Model

This chapter applies the basic arbitrage theorem in a single-period setting to develop a version of what is known as the capital asset pricing model (or CAPM).

A PRICING RELATION

The CAPM assumes that there are some agents whose holdings of securities do not depend on security prices. The aggregate short holdings of such agents, or equivalently the aggregate long holdings of all other agents, is known as the market portfolio.

Given the existence of a meaningful (positive and risky) market portfolio the CAPM assumes that payoffs of securities in some given units of value are linear functions of the payoff of this market portfolio. Then the payoff of security \( i \) is

\[
x_i = \alpha_i + \gamma_i x_a
\]

where \( \alpha_i \) and \( \gamma_i \) are constants and \( x_a \) is the payoff of the market portfolio \( a \).

Given some arbitrary probabilities the payoff of security \( i \) may be written as

\[
x_i = \mathbb{E}x_i + \gamma_i(x_a - \mathbb{E}x_a)
\]

where \( \mathbb{E} \) is the expectation operator under these probabilities.
Since the covariance of \( x_i \) with \( x_a \) is
\[
\text{cov}(x_i, x_a) = E(x_i - Ex_i)(x_a - Ex_a)
\]
\[
= E\gamma_i(x_a - Ex_a)^2
\]
\[
= \gamma_i E(x_a - Ex_a)^2
\]
\[
= \gamma_i \text{var}(x_a)
\]
where \( \text{var}(x_a) \) is the variance of \( x_a \), the parameter \( \gamma_i \) is given by
\[
\gamma_i = \text{cov}(x_i, x_a)/\text{var}(x_a).
\]

As this linear payoff relation applies for all securities it also applies for all linear combinations of securities, that is, for all portfolios. Thus if \( b \) is a portfolio then
\[
\alpha_b = \sum_i b_i x_i = \alpha_a + \gamma_b x_a.
\]

Since
\[
\text{cov}(x_b, x_a) = \text{cov}(\sum_i b_i x_i, x_a) = \sum_i b_i \text{cov}(x_i, x_a)
\]
we have
\[
\gamma_b = \sum_i b_i \text{cov}(x_i, x_a)/\text{var}(x_a) = \sum_i b_i \gamma_i = b \cdot \gamma.
\]

In particular, for the portfolio \( a \) we have
\[
\gamma_a = \text{cov}(x_a, x_a)/\text{var}(x_a) = \text{var}(x_a)/\text{var}(x_a) = 1.
\]

Choose some portfolio \( b \) such that
\[
\alpha_b = \sum_i b_i \alpha_i > 0
\]
and
\[
\gamma_b = \sum_i b_i \gamma_i = 0.
\]