This chapter continues the application of the basic arbitrage theorem in a many-period setting to value call and put options, and thus to value all derivative securities.

**CALLS AND PUTS**

We continue to consider contracts on a security whose payoff occurs on date $s$ only, the price of which security on date $t$ is denoted by $p_t$. We assume that the security is a limited liability security, so that $p_t$ is positive on each date $t$.

A European call option (with maturity $s$ and positive exercise price $q$), or a call, is a contract which gives its holder the right but not the obligation to buy the security on date $s$ for the exercise price $q$. Similarly, a put gives its holder the right to sell the security on date $s$ at the price $q$. The right involved in a call will be exercised if and only if the security price on date $s$ exceeds the exercise price, that is, if and only if

$$p_s > q,$$

while the right involved in a put will be exercised if and only if

$$p_s < q.$$

Equivalently, then, the call is a security which specifies the receipt on date $s$ of the amount $p_s - q$ if $p_s$ exceeds $q$ and a zero receipt otherwise, that is, which specifies the payoff

$$\max(p_s - q, 0)$$
on date $s$. Similarly, the put is a security which specifies the payoff

$$\max(q - p_s, 0)$$

on date $s$. The relation between the payoff of a call $c_s$ and the security price at maturity $p_s$ is illustrated in Figure 6.1; the corresponding relation for a put is illustrated in Figure 6.2.

We assume that there is a bond with unit payoff on date $s$, and also that on each date $t$ there is a bill with unit payoff on date $t + 1$ (on date $s$ the bill is 'instantaneous', its price being unity). Bond, bill, call, and put prices on date $t$ are denoted by $b_t, a_t, c_t,$ and $d_t$ respectively.

Applying the martingale property to the call on date 0 gives

$$c_0 = \hat{E}\delta\max(p_s - q, 0) = \hat{E}\max(\delta p_s - \delta q, 0)$$

where $\delta$ is the relevant discount factor and $\hat{E}$ the martingale expectation operator, so that

$$c_0 + \hat{E}\delta q = \hat{E}\max(\delta p_s, \delta q).$$

Applying the martingale property to the put gives

$$d_0 = \hat{E}\delta\max(q - p_s, 0) = \hat{E}\max(\delta q - \delta p_s, 0),$$

so that

$$d_0 + \hat{E}\delta p_s = \hat{E}\max(\delta q, \delta p_s).$$

It follows that

$$c_0 + \hat{E}\delta q = c_0 + q\hat{E}\delta = d_0 + \hat{E}\delta p_s.$$

Now applying the martingale property to the bond and the security gives

$$b_0 = \hat{E}\delta$$