Chapter 2

Elementary Matrix Algebra

2.1 The matrix notation

A matrix is a rectangular array of elements in rows and columns. Examples of matrices are

\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\quad
\begin{bmatrix}
a & b & c \\
c & c & b \\
2a & 2c & b
\end{bmatrix}
\quad
\begin{bmatrix}
x_{11} & x_{12} & x_{13} \\
x_{21} & x_{22} & x_{23}
\end{bmatrix}
\]

which are rectangular arrays with 2 rows and 2 columns, 3 rows and 3 columns and 2 rows and 3 columns respectively. The order of a matrix is the number of rows and the number of columns, so that in the above examples the orders are (2 \times 2) or (2 by 2), (3 \times 3), and (2 \times 3) respectively. The elements, which may be numbers or constants or variables, are enclosed between square brackets to signify that they must be considered as a whole and not individually. In contrast to the determinants of Chapter 1, a matrix does not have a single numerical value.

A matrix is often denoted by a single letter in bold-face type and a general matrix of order \( m \times n \) is written as

\[
X = \begin{bmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{bmatrix}
\]

where the subscripts identify the row and column in which the element is located. For example

- \( x_{12} \) is the element in row 1, column 2
- \( x_{34} \) is the element in row 3, column 4

and

\( x_{ij} \) is the element in row \( i \), column \( j \).
A square matrix has an equal number of rows and columns and in the general case has order \((n \times n)\). If a matrix has only one row it is referred to as a row vector and similarly a matrix with only one column is referred to as a column vector. Examples of these are

\[
A = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} \quad \text{is a square matrix of order} \ (2 \times 2)
\]

\[
B = \begin{bmatrix}
3 & 4 & 5 & 1
\end{bmatrix} \quad \text{is a row vector of order} \ (1 \times 4)
\]

\[
C = \begin{bmatrix}
c_{11} \\
c_{21}
\end{bmatrix} \quad \text{is a column vector of order} \ (2 \times 1)
\]

### 2.2 Elementary matrix operations

**Matrix Equality**

Two matrices are equal if, and only if, the corresponding elements of each matrix are all equal. For example,

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
4 \\
2
\end{bmatrix} \quad \text{if} \quad x_1 = 4 \quad \text{and} \quad x_2 = 2
\]

\[
\begin{bmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} \quad \text{if} \quad x_{11} = a_{11}, \quad x_{12} = a_{12} \\
x_{21} = a_{21}, \quad x_{22} = a_{22}
\]

and, in general

\[
\begin{bmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

that is, \(X = A\), if \(x_{ij} = a_{ij}\) for all \(i\) and all \(j\), where \(x_{ij}, a_{ij}\) represent the elements in the \(i\)th row and \(j\)th column of the two matrices respectively.

It follows that two matrices can be equal only if they have the same number of elements arranged in corresponding positions, i.e. they must be of the same order. In the above case, both \(X\) and \(A\) are \((m \times n)\).