SIMPLE ALGEBRAIC APPROXIMATIONS FOR THE EFFECTIVE ELASTIC MODULI OF A CUBIC ARRAY OF SPHERES

Israel Cohen and David J. Bergman
School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978 Israel
Israel Cohen <israelc@post.tau.ac.il>
David Bergman <bergman@post.tau.ac.il>

Abstract The method of elastostatic resonances [1] is applied to the three-dimensional problem of nonoverlapping spherical inclusions arranged in a cubic array in order to calculate the effective elastic moduli. The leading order in this systematic perturbation expansion is related to the Clausius-Mossotti (CM) approximation of electrostatics. It takes into account the dipole-dipole interaction between strain fields of different inclusions, and makes use of the concept of the the local Lorentz field. The derived CM-type approximations are in the form of simple algebraic expressions [2]. They provide accurate results at low volume fractions of the inclusions and are good estimates at moderate volume fractions even when the contrast is high. The expression for the bulk modulus turns out to be identical to one of the Hashin-Shtrikman bounds.

Keywords: Mechanical properties, effective elastic moduli, Clausius-Mossotti.

1. Introduction

Kantor and Bergman [1] introduced an approach to the problem of calculating the effective elastic stiffness tensor $C^{(e)}$ of two-component composite materials with a specified microstructure. This approach, which was based on a calculation of elastostatic resonances of the system, was applied to composites in the form of regular two-dimensional (2D) arrays of circular-cylindrical inclusions and to 3D periodic arrays of differently oriented circular-cylindrical inclusions with cubic symmetry [3]. We apply the same approach to a 3D model of cubic arrays of spherical inclusions of isotropic material $C^{(1)}(\kappa_1, \mu_1)$ (component 1) embedded in an isotropic host $C^{(2)}(\kappa_2, \mu_2)$ (component 2) ($\kappa$ is the bulk modulus, $\mu$ is the shear modulus). This simple model is already a very difficult problem to solve, mainly because the eigenstates (i.e., elastostatic resonances) of an isolated spherical inclusion are not known. In fact, we have computed only a few of them—the dipole eigenstates. This turns out to be a
useful exercise because these states are often responsible for the dominant part of the interaction between distortions of different inclusions, in analogy with electrostatic problems. The results obtained for the macroscopic elastic moduli are in the form of simple algebraic expressions, similar both in form and in spirit to the Clausius-Mossotti (CM) expression for macroscopic dielectric constants of such composites. Corrections to these expressions begin at order that is not less than $p^{11/3}$.

2. Summary of the underlying theory

The approach begins by introducing a somewhat generalized form of the original problem and replacing the $C^{(1)}$ material by a different material $C^{(1)}(s)$, where $C^{(1)}(s) = C^{(2)} - (1/s)\delta C$, and $\delta C \equiv C^{(2)} - C^{(1)}$. This replacement also makes $C(\varepsilon)$ a function of the parameter $s$. When $s$ lies in certain ranges the tensor $C^{(1)}(s)$ becomes unphysical, i.e., it ceases to be positive definite. The original problem is retrieved by setting $s = 1$. The strain tensor $\varepsilon(r)$ in such a composite material, the boundaries of which undergo the displacement $u_i = \varepsilon^{(0)}_{ij} x_j$, is the solution of the operator equation

$$\varepsilon = \varepsilon^{(0)} + \frac{1}{s} \hat{\Gamma} \varepsilon,$$  \hfill (1)

where $\varepsilon^{(0)}_{ij}$ is any constant symmetric tensor and $\hat{\Gamma}$ is a linear integral operator related to the tensor Green function of the problem. A scalar product of two second-rank symmetric tensor fields $\varepsilon, \sigma$ is defined ($\ast$ denotes complex conjugation): $\langle \varepsilon | \sigma \rangle \equiv \int dV \theta_1(r) \varepsilon^*_{ij}(r) \sigma_{ij}(r)$, where $\theta_1(r)$ is equal to 1 only inside component 1, and vanishes elsewhere. Using this definition, the effective elastic tensor can be calculated from a related function $F(s)$:

$$F(s) = \varepsilon^{(0)} C^{(2)} \varepsilon^{(0)} - \varepsilon^{(0)} C(\varepsilon) \varepsilon^{(0)} = \langle \varepsilon^{(0)} | \delta C | \varepsilon^{(0)} \rangle,$$

where summation over tensorial indices is implied. Under the scalar product the operator $\hat{\Gamma}$ is not Hermitian, and thus has different right and left eigenstates. However, if $|\varepsilon^{(n)}\rangle$ is a right eigenstate with eigenvalue $s_n$, then $\langle \varepsilon^{(n)} | \delta C | \varepsilon^{(m)} \rangle = 0$ for $n \neq m$. It can be proven that the eigenvalues $s_n$ are all real, and that the eigenstates are normalizable with a norm $\| \varepsilon \|$ defined to be $\| \varepsilon \| \equiv \langle \varepsilon | \varepsilon \rangle^{1/2}$. The eigenstate $|\varepsilon^{(n)}\rangle$ represents an elastostatic resonance of the sample, i.e., a state where the sample is internally deformed and strained even though the boundaries are undeformed. Obviously, such resonances can occur only at unphysical values of $C^{(1)}(s)$. The way to proceed is to substitute the formal solution of Eq. (1) for $\varepsilon$ in the scalar product in the expression for $F(s)$, and then expand it using the bi-orthogonal set $\varepsilon^{(n)}$. This way, a spectral