Several approximations were already carried out in earlier chapters. In this respect, for example in Chapter 7, we have considered relativistic corrections to the energy levels of the hydrogen atom, also the non-relativistic Lamb shift, and treated the atom in external electromagnetic fields. In Chapter 8, we have discussed the validity of the exponential law in a two-level system (§8.1) and related approximations involved. In the same chapter, radiation loss in spin precession was studied (§8.4), a computation of the anomalous magnetic moment of the electron was made (§8.5), the problem of quantum decoherence by the environment was considered (§8.7, §8.9) and the so-called geometric phase in the adiabatic approximating regime was investigated (§8.13). Some approximations are also given in Chapter 15 on quantum scattering, just to mention a few of the applications of approximation methods. The present chapter supplements these studies by investigating the nature of several approximation procedures, some of which are related to the above applications in other chapters. Accordingly, the latter material may be read in conjunction with the present one.

Sections 12.1, 12.2 deal with conventional time-independent perturbation theories, followed by one on variational methods. High-order perturbations and related divergent series as applied to an anharmonic oscillator potential is the subject matter of §12.4. In §12.5, we study the so-called semi-classical WKB approximation. Time-dependent perturbation theory is treated in §12.5 dealing, in particular, with the sudden and adiabatic approximations in which a Hamiltonian may change rapidly in time in a very short time interval and in the other extreme a Hamiltonian may change very slowly during a long time span, respectively. In the last section, we study, in the density operator formalism, the response of a system, into consideration, to another system, such as the environment. In this section, we derive the master equation describing the dynamics of the reduced density operator of the system of interest after having traced the density operator of the combined two systems over the variables of the other one.
12.1 Non-Degenerate Perturbation Theory

Consider a Hamiltonian $H^0$ which, as part of its spectrum, has a discrete non-degenerate one with eigenvalue equation

$$H^0 |n\rangle_0 = E^0_n |n\rangle_0$$  (12.1.1)

where the $E^0_n$ are non-degenerate and the eigenvectors $|n\rangle_0$ are orthonormal. We add to $H^0$ a term $H_1$, referred to as a perturbation, which is in some sense small in comparison to $H^0$, thus introducing the Hamiltonian

$$H = H^0 + H_1.$$  (12.1.2)

We suppose that for the new system, we have an eigenvalue equation

$$H |n\rangle = E_n |n\rangle$$  (12.1.3)

and that $E_n$ is near $E^0_n$ for a given quantum number $n$. The shift in energy due to the addition of the perturbation $H_1$ to $H^0$ is defined by

$$\Delta E_n = E_n - E^0_n$$  (12.1.4)

and we set

$$|n\rangle = |n\rangle_0 + |n\rangle'.$$  (12.1.5)

Upon multiplying (12.1.3) from the left by $0\langle n|$ and using (12.1.1)–(12.1.5), we obtain for the energy shift the expression

$$\Delta E_n = \frac{0\langle n|H_1|n\rangle_0}{0\langle n|n\rangle}.$$  (12.1.6)

This is invariant under phase transformations

$$|n\rangle \rightarrow |n\rangle e^{i\gamma}$$  (12.1.7)

as the phase factor cancels out from the numerator and the denominator in (12.1.6).

To first order in $H_1$, we may replace $|n\rangle$ by $|n\rangle_0$ in (12.1.6), to get for the energy shift

$$\Delta E^1_n = 0\langle n|H_1|n\rangle_0$$  (12.1.8)

or

$$E_n \simeq E^0_n + E^1_n$$  (12.1.9)

with

$$E^1_n = 0\langle n|H_1|n\rangle_0.$$  (12.1.10)

To first order, we write

$$|n\rangle \simeq |n\rangle_0 + |n\rangle_1$$  (12.1.11)