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Information in Multirate Systems

7.1 Introduction

Recall from Chapter 2 that all the statistical information about a Gaussian WSS random signal $x(n)$ is contained in its power spectrum $P_x(e^{j\omega})$ or, equivalently, in its autocorrelation sequence $R_x(k)$. This is because if we know $P_x(e^{j\omega})$, we can calculate the probability density functions that govern the statistical dependence among any number of samples of $x(n)$.

Assume that a WSS signal $x(n)$ is not available for statistical experiment directly but we have access some multi-rate measurement signals $v_i(n)$ obtained indirectly via a multirate measurement system (Fig. 7.1). One may pose questions like these: How much statistical information about $x(n)$ can be gained if we calculate statistical properties of $v_0(n)$? Which signal, $v_2(n)$ or $v_3(n)$, gives more statistical information about $x(n)$? If we know statistical...
properties of \(w_t(n)\), how much more information about \(x(n)\) will be gained by doing statistical experiments on \(v_t(n)\)?

To answer questions of the type mentioned above, we need to establish a quantitative measure of statistical information gained about \(x(n)\) by statistical experiments performed on the multirate measurements \(v_t(n)\). The purpose of this chapter is to introduce one such measure.

The material in this chapter are based on the paper by Jahromi et al. (2004a).

### 7.2 Information as distance from uniform spectrum

Consider the multirate filter bank in Fig. (7.2) and assume that we know a priori that \(x(n)\) is a regular WSS random signal with zero mean and unit variance\(^1\).

We say that the state of our knowledge regarding the statistics of \(x(n)\) is “complete ignorance” if all we know about the statistics of \(x(n)\) is the assumptions stated in the above paragraph. By definition, we assign the value zero to the quantity of information associated with this state of knowledge.

When in the state of complete ignorance, we assume based on the principle of Maximum Entropy that the power spectrum of \(x(n)\) is a white (constant) spectrum with unit variance. In other words, in the state of complete ignorance we assume that \(P_x = \bar{P}_x\) where

\[
\bar{P}_x(e^{j\omega}) \triangleq 1, \ \omega \in [-\pi, \pi]. \tag{7.1}
\]

We say that the state of our knowledge regarding the statistics of \(x(n)\) is “complete information” if we know \(P_x(e^{j\omega})\) exactly. By definition, the amount of information associated with this state of knowledge is given by

\[
I(x) \triangleq D(P_x \parallel \bar{P}_x), \tag{7.2}
\]

where \(D(P_x \parallel \bar{P}_x)\) is the Kullback-Leibler divergence of \(P_x\) from \(\bar{P}_x\).

Obviously, “complete information” and “complete ignorance” are two extreme states of knowledge with regards to the statistics of \(x(n)\). These states, therefore, mark the upper and lower limits on the information scale (Fig. 7.2) which represents the quantity of information that can be gained via multirate measurements.

Recall from Chapter 2 that the entropy rate \(H(x)\) of a Gaussian random process \(x(n)\) is given by

\[^1\] The assumption that the input signal’s variance is normalized to one is necessary to avoid the ambiguity caused by the scale-dependence of our information measure. This is a regrettable limitation but is shared by all entropy-based information measures.