CHAPTER 4
TRANSFER FUNCTION APPROACHES

4.1. RESIDENCE TIME DISTRIBUTIONS

In many problems of interest to modelers, the goal is to relate system output to inputs. In some applications of solute transport through porous media, for example, one is often interested in the effluent concentration out of a soil column as a function of time for a given influent concentration. This can be very conveniently accomplished by the residence time distribution (RTD) theory originally proposed by Danckwerts (1953). The concept of RTDs is not restricted to soil columns alone and can be used in a wide variety of flow and solute transport problems. Provided that the process or system being modeled can be treated in a linear fashion, the RTD theory provides a way of characterizing the system without having to know explicitly about the inner mechanics of the system in any detail. In the context of solute moving through a soil column, this would imply that the exact path taken by each solute particle along with the local velocities do not need to be characterized in a detailed manner.

Consider that a conservative solute of mass \( M \) is released instantaneously at time \( t = 0 \) from a source location. The influent flux concentration is mathematically \( c_{in}(t) = \frac{M}{Q_{in}(t)} \delta(t) \), where \( Q_{in}(t) \) is the volumetric flowrate at the inflow end, and \( \delta(t) \) is the Dirac-delta function. If the effluent flux concentration (mass per unit volume of liquid) is described by \( c_f(t) \), from mass continuity we require that

\[
\int_{0}^{\infty} c_{in}(t) dt = M = \int_{0}^{\infty} Q_{out}(t) c_f(t) dt
\]

(4.1.1)

where \( Q_{out}(t) \) is the volumetric flow rate at the outlet. For cases of steady flows, the inflow and outflow rates are a constant, \( Q \). If, after appropriate normalization, the effluent concentration function is treated as a probability density function, then the machinery of random variables can be applied to these problems. We first define

\[
f_T(t) = \frac{Q}{M} c(t)
\]

(4.1.2)

In this example, since all the solute particles were introduced instantaneously at \( t = 0 \), \( f_T(t) \) is a representation of how long solute particles stay within the soil
column (system). Indeed, if $T$ is a random variable denoting the travel time from the source to the receptor location (i.e. the residence time in the system), then $f_T(t)$ may be viewed as the pdf of travel times (see Figure 4.1.1). On the other hand, if the inlet concentration is held at a constant value $c_0$ (i.e. $c_i(t) = c_0H(t)$, where $H(t)$ is the Heaviside step function), then the effluent curve can be thought of as cumulative distribution function after normalization.

Since the effluent concentration function under a Dirac input can be thought of as a probability density function, the center of mass of the effluent curve under instantaneous injection of solute at the inlet is the expected value of the travel time, i.e.

$$E[T] = \int_0^\infty t \, f_T(t) \, dt = \int_0^\infty [1 - F(t)] \, dt$$  \hspace{1cm} (4.1.4)$$

This conceptualization allows us to define the displaced pore volume $V_p$ as

$$V_p = QE[T]$$  \hspace{1cm} (4.1.5)$$

If the velocity field is transient, then the travel time pdf would need to be conditioned on the inlet time – the time at which the solute particle entered the system. The above definitions are valid for steady flows.

### 4.2. MODELS OF SOLUTE TRANSPORT

The question of the actual form of $f_T(t)$ can be addressed in two ways. The first method would be to conduct experiments with measured inputs and outputs, and $f_T(t)$ would be determined from experimental observations. The second approach