Chapter 10

RING-SHAPED NON-SPHERICAL OSCILLATOR

1. Introduction

Up to now, the algebraic method has been a subject of the interest in various fields of physics. With the factorization method, we have established the ladder operators for quantum systems with some important potentials and constructed suitable Lie algebras. Recently, the quantum system for the ring-shaped non-spherical oscillator has been studied [463]. The purpose of this Chapter is to study its hidden symmetry.

This Chapter is organized as follows. In Section 2 we derive the eigenvalues and eigenfunctions of this system. Section 3 is devoted to establishing the ladder operators directly from the radial wave functions. In Section 4 we shall realize the dynamic group and obtain the analytical matrix elements of physical functions $\rho$ and $\rho \frac{d}{d\rho}$ with $\rho = r^2$. Some conclusions are given in Section 5.

2. Exact solutions

In this section we study the exact solutions of the Schrödinger equation with the ring-shaped non-spherical oscillator, which are necessary for constructing the ladder operators. Consider the Schrödinger equation with a potential $V(r)$

$$\frac{-\hbar^2}{2m} \nabla^2 \Psi(r) + V(r) \Psi(r) = E \Psi(r),$$  \hspace{1cm} (10.1)

where $m$ is the mass of the particle. In the present work, the potential $V(r)$ is taken as a ring-shaped non-spherical oscillator,

$$V(r, \theta) = \frac{1}{2} m \omega^2 r^2 + \frac{\hbar^2}{2m} \left( \frac{\alpha}{r^2} + \frac{\beta}{r^2 \sin^2 \theta} \right),$$  \hspace{1cm} (10.2)

where the $\omega$ is the frequency; $\alpha$ and $\beta$ are two dimensionless parameters.
For simplicity, the natural units $\hbar = m = \omega = 1$ are employed. Due to the symmetry of the potential, we take the wave functions with the form

$$\Psi(r, \theta, \varphi) = \frac{R(r)}{r} \Theta(\theta) e^{\pm im\varphi}, \quad m = 0, 1, 2, \ldots$$  \hfill (10.3)

Substitution of this into Eq. (10.1) allows us to obtain the radial equation and the angular one as follows:

$$\frac{d^2 R(r)}{dr^2} + \left(2E - r^2 - \frac{\alpha + \lambda}{r^2}\right) R(r) = 0,$$  \hfill (10.4)

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta(\theta)}{d\theta}\right) + \left(\lambda - \frac{\beta + m^2}{\sin^2 \theta}\right) \Theta(\theta) = 0,$$  \hfill (10.5)

with the constant $\lambda$ to be determined below.

Introduce

$$\xi = \sqrt{\beta + m^2}, \quad \lambda = \ell'(\ell' + 1), \quad x = \cos(\theta).$$  \hfill (10.6)

Substitution of Eq. (10.6) into Eq. (10.5) leads to

$$(1 - x^2) \frac{d^2 \Theta(x)}{dx^2} - 2x \frac{d\Theta(x)}{dx} + \left[\ell' (\ell' + 1) - \frac{\xi^2}{1 - x^2}\right] \Theta(x) = 0,$$  \hfill (10.7)

which is called the so-called universal associated Legendre differential equation. Its normalized solutions can be obtained as

$$\Theta_{\ell'\xi}(\theta) = \sqrt{\frac{(2\ell' + 1)}{2}} \frac{(\ell' - \xi)!}{\Gamma(\ell' + \xi + 1)} (\sin \theta)^{\xi} \cdot \left[\frac{\ell' - \xi}{2}\right] \sum_{v=0} \frac{(-1)^v \Gamma(2\ell' - 2v + 1)}{2^{\ell' + 1} v!(\ell' - \xi - 2v)! \Gamma(\ell' - v + 1)} (\cos \theta)^{\ell' - \xi - 2v},$$  \hfill (10.8)

with

$$\ell' = \xi + k, \quad k = 0, 1, 2, \ldots$$  \hfill (10.9)

Substitution of this into Eq. (10.4) allows us to obtain

$$\frac{d^2 R(r)}{dr^2} + \left(2E - r^2 - \frac{L(L + 1)}{r^2}\right) R(r) = 0,$$  \hfill (10.10)

with

$$L \equiv \frac{1}{2} \left[\sqrt{1 + 4 \left[\alpha + (\sqrt{\beta + m^2 + k}) (\sqrt{\beta + m^2 + k + 1})\right]} - 1\right].$$  \hfill (10.11)